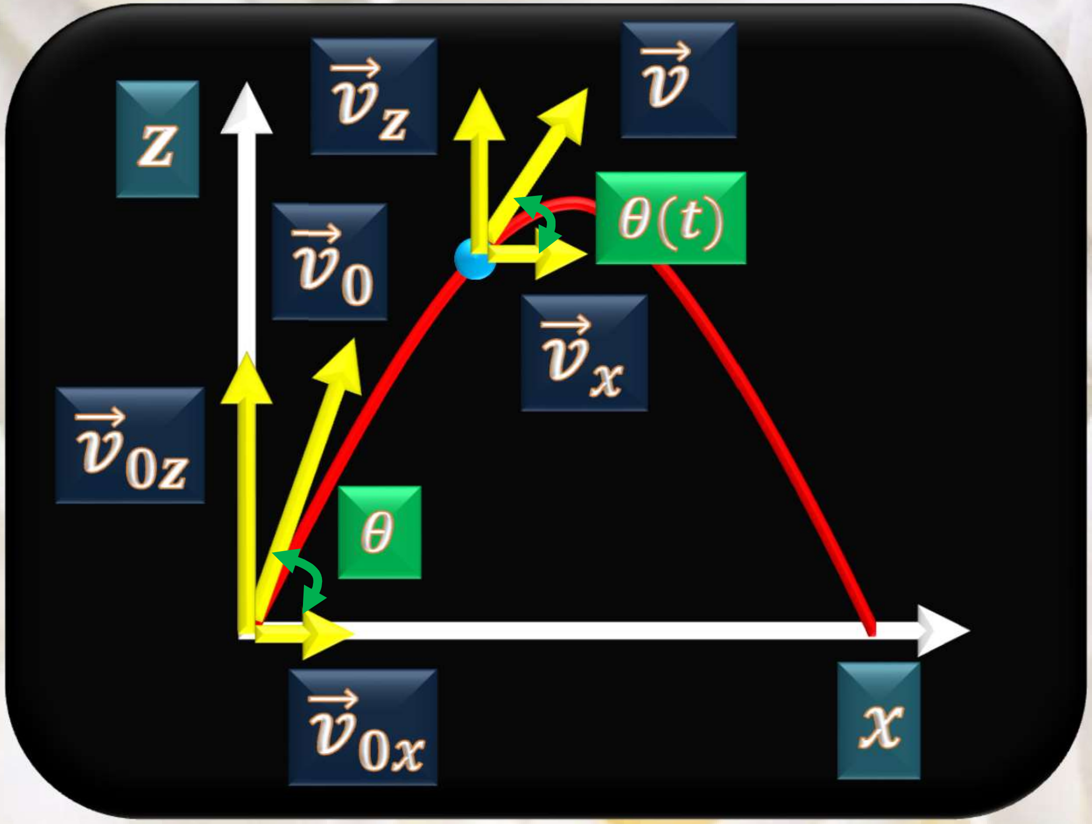
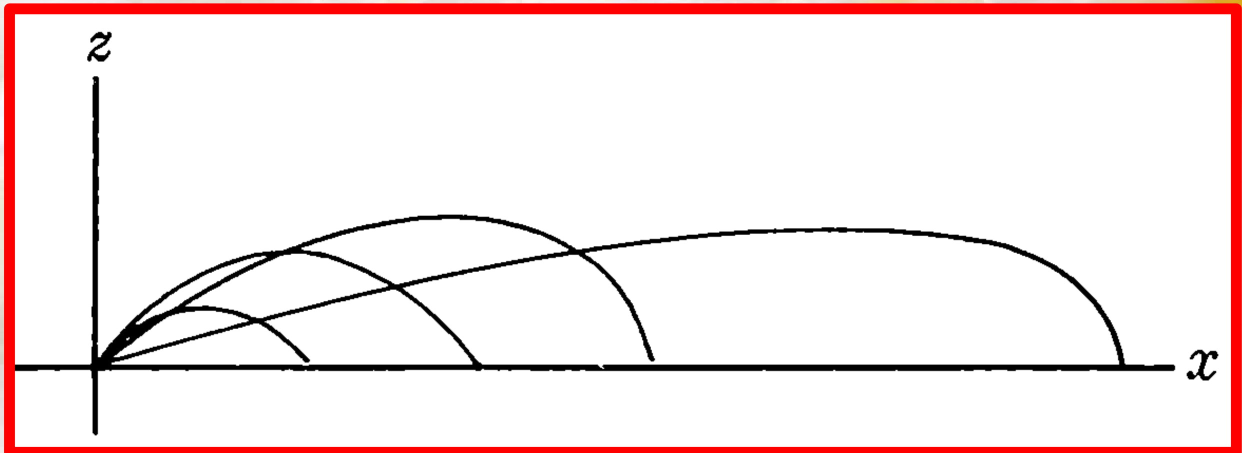
حرکت پرتابی در حضور مقاومت هوا





$$\vec{v}_0 = v_{0x}\hat{\imath} + v_{0z}\hat{k}$$

$$v_{0x} = v_0 cos\theta$$

$$m\frac{dv_x}{dt} = -bv_x$$

$$\frac{dv_x}{v_x} = -\frac{b}{m}dt$$

$$lnv_x - lnv_{0x} = -\frac{b}{m}t$$

$$ln\left(\frac{v_x}{v_{0x}}\right) = -\frac{b}{m}t$$

$$\frac{v_x}{v_{0x}} = exp\left(-\frac{b}{m}t\right)$$

$$v_x = v_{0x}exp\left(-\frac{b}{m}t\right)$$

$$v_x = v_0 cos\theta e^{-bt/m}$$

$$v_x \approx v_0 cos\theta \left(1 - \frac{bt}{m}\right)$$

$$v_x \approx v_0 cos\theta - \frac{b v_0 cos\theta}{m} t$$

$$v_x = \frac{dx}{dt} = v_{0x}exp\left(-\frac{b}{m}t\right)$$

$$dx = v_{0x}exp\left(-\frac{b}{m}t\right)dt$$

$$x = v_{0x}\left(-\frac{m}{b}\right)exp\left(-\frac{b}{m}t\right)\Big|_0^t$$

$$x = \frac{m v_{0x}}{b}\left(1 - exp\left(-\frac{b}{m}t\right)\right)$$

$$x = \frac{m v_0 cos\theta}{b}\left(1 - e^{-bt/m}\right)$$

$$x \approx \frac{m v_0 cos\theta}{b}\left(1 - 1 + \frac{bt}{m} - \frac{b^2t^2}{2m^2}\right)$$

$$x \approx v_0 cos\theta t - \frac{1}{2}\frac{b v_0 cos\theta}{m}t^2$$

$$v_{0z} = v_0 sin\theta$$

$$m\frac{dv_z}{dt} = -mg - bv_z$$

$$\frac{dv_z}{mg + bv_z} = -\frac{1}{m}dt$$

$$mg + bv_z = u$$

$$bdv_z = du$$

$$\frac{\frac{1}{b}du}{u} = -\frac{1}{m}dt$$

$$\frac{1}{b}lnu = -\frac{1}{m}t$$

$$ln(mg + bv_z)\Big|_{v_{0z}}^{v_z} = -\frac{b}{m}t$$

$$ln\left(\frac{mg + bv_z}{mg + bv_{0z}}\right) = -\frac{b}{m}t$$

$$\frac{mg + bv_z}{mg + bv_{0z}} = e^{-bt/m}$$

$$mg + bv_z = (mg + bv_{0z})e^{-bt/m}$$

$$v_z = \left(\frac{mg}{b} + v_{0z}\right)e^{-bt/m} - \frac{mg}{b}$$

$$v_z = \left(\frac{mg}{b} + v_0 sin\theta\right)e^{-b\ /m} - \frac{mg}{b}$$

$$v_z = \left(\frac{mg}{b} + v_0 sin\theta\right) e^{-bt/m} - \frac{mg}{b}$$

$$v_z \approx \left(\frac{mg}{b} + v_0 sin\theta\right)\left(1 - \frac{bt}{m} + \frac{b^2t^2}{2m^2}\right) - \frac{mg}{b}$$

$$v_z \approx \frac{mg}{b} + v_0 sin\theta - gt - \frac{bv_0 sin\theta t}{m} + \frac{bg}{2m}t^2 - \frac{mg}{b}$$

$$v_z \approx -gt + v_0 sin\theta + \frac{bt}{m}\left(-v_0 sin\theta + \frac{1}{2}gt\right)$$

$$\frac{dz}{dt} = v_z = \left(\frac{mg}{b} + v_{0z}\right) e^{-bt/m} - \frac{mg}{b}$$

$$dz = \left(\frac{mg}{b} + v_{0z}\right) e^{-bt/m} dt - \frac{mg}{b} dt$$

$$z = \left(\frac{mg}{b} + v_{0z}\right)\left(-\frac{m}{b}\right) e^{-bt/m}\Big|_0^t - \frac{mg}{b} t\Big|_0^t$$

$$z = \left(\frac{m^2 g}{b^2} + \frac{m v_{0z}}{b}\right)\left(1 - e^{-\frac{bt}{m}}\right) - \frac{mg}{b} t$$

$$z = \left(\frac{m^2 g}{b^2} + \frac{m v_0 sin\theta}{b}\right)\left(1 - e^{-bt/m}\right) - \frac{mg}{b} t$$

$$z=\left(\frac{m^2g}{b^2}+\frac{mv_0sin\theta}{b}\right)\left(1-e^{-bt/m}\right)-\frac{mg}{b}t$$

$$z\approx\left(\frac{m^2g}{b^2}+\frac{mv_0sin\theta}{b}\right)\left(\frac{bt}{m}-\frac{b^2t^2}{2m^2}+\frac{b^3t^3}{6m^3}\right)-\frac{mg}{b}t$$

$$z\approx\frac{mgt}{b}-\frac{1}{2}gt^2+\frac{gb}{6m}t^3+v_0sin\theta t-\frac{bv_0sin\theta}{2m}t^2-\frac{mg}{b}t$$

$$z\approx-\frac{1}{2}gt^2+v_0sin\theta t-\frac{bv_0sin\theta}{2m}t^2+\frac{gb}{6m}t^3$$

$$z\approx-\frac{1}{2}gt^2+v_0sin\theta t-\frac{bt}{2m}\left(v_0sin\theta t-\frac{1}{3}gt^2\right)$$

معادله مستقل از زمان به صورت زیر قابل محاسبه است:

$$x = \frac{mv_0 cos\theta}{b}\left(1 - e^{-bt/m}\right)$$

$$1 - e^{-bt/m} = \frac{bx}{mv_0 cos\theta}$$

$$e^{-bt/m} = 1 - \frac{bx}{mv_0 cos\theta}$$

$$-\frac{bt}{m} = ln\left(1 - \frac{bx}{mv_0 cos\theta}\right)$$

$$z = \left(\frac{m^2 g}{b^2} + \frac{mv_0 sin\theta}{b}\right)\left(1 - e^{-bt/m}\right) - \frac{mg}{b}t$$

$$z = \left(\frac{m^2 g}{b^2} + \frac{m v_0 sin\theta}{b}\right)\frac{bx}{m v_0 cos\theta} + \frac{m^2 g}{b^2} ln\left(1 - \frac{bx}{m v_0 cos\theta}\right)$$

$$z = \left(\frac{mg}{b v_0 cos\theta} + tan\theta\right)x - \frac{m^2 g}{b^2} ln\left(\frac{m v_0 cos\theta}{m v_0 cos\theta - bx}\right)$$

$$z = \left(\frac{mg}{b v_0 cos\theta} + tan\theta\right)x + \frac{m^2 g}{b^2} ln\left(1 - \frac{bx}{m v_0 cos\theta}\right)$$

$$z \approx \left(\frac{mg}{b v_0 cos\theta} + tan\theta\right)x + \frac{m^2 g}{b^2}\left(-\frac{bx}{m v_0 cos\theta} - \frac{b^2 x^2}{2 m^2 v_0^2 cos^2\theta} - \frac{b^3 x^3}{3 m^3 v_0^3 cos^3\theta}\right)$$

$$z \approx \frac{mgx}{b v_0 cos\theta} + x tan\theta - \frac{mgx}{b v_0 cos\theta} - \frac{g x^2}{2 v_0^2 cos^2\theta} - \frac{b g x^3}{3 m v_0^3 cos^3\theta}$$

$$z \approx x tan\theta - \frac{g}{2 v_0^2 cos^2\theta} x^2 - \frac{bg}{3 m v_0^3 cos^3\theta} x^3$$

در نقطه اوج مولفه قایم سرعت صفر است، پس

$$v_z=\left(\frac{mg}{b}+v_0 sin\theta\right)e^{-bt/m}-\frac{mg}{b}=0$$

$$e^{bt/m}=\frac{b}{mg}\left(\frac{mg}{b}+v_0 sin\theta\right)$$

$$\frac{bt}{m}=ln\left[\frac{b}{mg}\left(\frac{mg}{b}+v_0 sin\theta\right)\right]$$

$$t=\frac{m}{b}ln\left(1+\frac{v_0 b}{mg}sin\theta\right)$$

$$ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$$

$$t=\frac{m}{b}\left(\frac{v_0 b}{mg}sin\theta-\frac{1}{2}\left(\frac{v_0 b}{mg}sin\theta\right)^2+\cdots\right)$$

$$t=\frac{v_0 sin\theta}{g}-\frac{1}{2}\frac{v_0^2 b sin^2\theta}{mg^2}+\cdots$$

زمان اثابت به زمین:

$$z = \left(\frac{m^2 g}{b^2} + \frac{m v_0 sin\theta}{b}\right)\left(1 - e^{-bt/m}\right) - \frac{mg}{b}t = 0$$

اما این معادله به روش تحلیلی قابل حل نیست، لذا آن را به صورت تقریبی حل می کنیم:

$$z \approx -\frac{1}{2}gt^2 + v_0 sin\theta t - \frac{bt}{2m}\left(v_0 sin\theta t - \frac{1}{3}gt^2\right) = 0$$

$$t = 0$$

که همان لحظه پرتاب است

$$-\frac{1}{2}gt + v_0 sin\theta - \frac{b}{2m}\left(v_0 sin\theta t - \frac{1}{3}gt^2\right) \approx 0$$

که یک معادله درجه دوم است، که پس از مرتب کردن می توان آن را به صورت زیر نوشت

$$\frac{bg}{6m}t^2-\frac{1}{2}\left(g+\frac{bv_0 sin\theta}{2m}\right)t+v_0 sin\theta\approx 0$$

$$-\frac{1}{2}gt+v_0 sin\theta\approx 0$$

$$t\approx\frac{2v_0 sin\theta}{g}$$

که اگر b=0 آنگاه به دو برابر زمان اوج می رسیم

$$t\approx\frac{\frac{1}{2}\left(g+\frac{bv_0 sin\theta}{2m}\right)\pm\sqrt{\left(\frac{1}{2}\left(g+\frac{bv_0 sin\theta}{2m}\right)\right)^2-4\frac{bg}{6m}v_0 sin\theta}}{2\frac{bg}{6m}}$$

$$t\approx\frac{\frac{1}{2}\left(g+\frac{bv_0 sin\theta}{2m}\right)\pm\sqrt{\frac{1}{4}\left(g^2+\frac{bg}{m}v_0 sin\theta+\frac{b^2}{4m^2}v_0^2 sin^2\theta\right)-\frac{2bg}{3m}v_0 sin\theta}}{\frac{bg}{3m}}$$

$$t\approx\frac{\frac{1}{2}\left(g+\frac{bv_0 sin\theta}{2m}\right)-\sqrt{\frac{1}{4}g^2-\frac{5bg}{12m}v_0 sin\theta+\frac{b^2}{16m^2}v_0^2 sin^2\theta}}{\frac{bg}{3m}}$$

$$t \approx \frac{\frac{1}{2}\left(g+\frac{bv_0 sin\theta}{2m}\right)-\frac{1}{2}g\sqrt{1-\frac{5b}{3mg}v_0 sin\theta+\frac{b^2}{4m^2g^2}v_0^2 sin^2\theta}}{\frac{bg}{3m}}$$

$$\sqrt{1-x}=1-\frac{1}{2}x+\cdots$$

$$t \approx \frac{\frac{1}{2}\left(g+\frac{bv_0 sin\theta}{2m}\right)-\frac{1}{2}g\left(1-\frac{1}{2}\left(\frac{5b}{3mg}v_0 sin\theta-\frac{b^2}{4m^2g^2}v_0^2 sin^2\theta\right)\right)}{\frac{bg}{3m}}$$

$$t \approx \frac{\frac{1}{2}g+\frac{bv_0 sin\theta}{4m}-\frac{1}{2}g+\frac{5bv_0 sin\theta}{12m}-\frac{b^2}{8m^2g}v_0^2 sin^2\theta}{\frac{bg}{3m}}$$

$$t \approx \frac{\frac{8bv_0 sin\theta}{12m}-\frac{b^2}{8m^2g}v_0^2 sin^2\theta}{\frac{bg}{3m}}$$

$$t \approx \frac{2v_0 sin\theta}{g}-\frac{3b}{8mg^2}v_0^2 sin^2\theta$$

که 2 برابر زمان اوج نیست

برد پرتابه به صورت زیر قابل محاسبه است

$$z \approx x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x^2 - \frac{bg}{3mv_0^3\cos^3\theta}x^3 = 0$$

$$x = 0$$

که همان نقطه شروع حرکت است

$$\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x - \frac{bg}{3mv_0^3\cos^3\theta}x^2 \approx 0$$

$$x \approx \frac{\frac{g}{2v_0^2\cos^2\theta} \pm \sqrt{\left(\frac{g}{2v_0^2\cos^2\theta}\right)^2 + \frac{4bg\sin\theta}{3mv_0^3\cos^4\theta}}}{-\frac{2bg}{3mv_0^3\cos^3\theta}}$$

$$x \approx \frac{\dfrac{g}{2v_0^2cos^2\theta} \pm \dfrac{g}{2v_0^2cos^2\theta}\sqrt{1+\dfrac{16bv_0sin\theta}{3gm}}}{-\dfrac{2bg}{3mv_0^3cos^3\theta}}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$x \approx \frac{\dfrac{g}{2v_0^2cos^2\theta} - \dfrac{g}{2v_0^2cos^2\theta}\left(1+\dfrac{1}{2}\dfrac{16bv_0sin\theta}{3gm} - \dfrac{1}{8}\dfrac{16^2b^2v_0^2sin^2\theta}{3^2g^2m^2}\right)}{-\dfrac{2bg}{3mv_0^3cos^3\theta}}$$

$$x \approx \frac{-\dfrac{g}{2v_0^2cos^2\theta}\left(\dfrac{8bv_0sin\theta}{3gm} - \dfrac{32b^2v_0^2sin^2\theta}{3^2g^2m^2}\right)}{-\dfrac{2bg}{3mv_0^3cos^3\theta}}$$

$$x \approx \frac{\frac{g}{2v_0^2 cos^2\theta}\left(\frac{8bv_0 sin\theta}{3gm} - \frac{32b^2 v_0^2 sin^2\theta}{3^2 g^2 m^2}\right)}{\frac{2bg}{3mv_0^3 cos^3\theta}}$$

$$x \approx \frac{\frac{4bsin\theta}{3mv_0 cos^2\theta} - \frac{16b^2 sin^2\theta}{3^2 gm^2 cos^2\theta}}{\frac{2bg}{3mv_0^3 cos^3\theta}}$$

$$x \approx \frac{2v_0^2 sin\theta cos\theta}{g} - \frac{8bv_0^3 sin^2\theta cos\theta}{3mg^2}$$

$$x \approx \frac{v_0^2 sin2\theta}{g} - \frac{4bv_0^3 sin2\theta sin\theta}{3mg^2}$$

محاسبه ارتفاع اوج

$$z = \left(\frac{mg}{bv_0 cos\theta} + tan\theta\right)x + \frac{m^2 g}{b^2} ln\left(1 - \frac{bx}{mv_0 cos\theta}\right)$$

$$\frac{dz}{dx} = \left(\frac{mg}{bv_0 cos\theta} + tan\theta\right) - \frac{\frac{m^2 g}{b^2}\frac{b}{mv_0 cos\theta}}{1 - \frac{bx}{mv_0 cos\theta}} = 0$$

$$1 - \frac{bx}{mv_0 cos\theta} = \frac{\frac{m^2 g}{b^2}\frac{b}{mv_0 cos\theta}}{\frac{mg}{bv_0 cos\theta} + tan\theta}$$

$$x = \left(1 - \frac{\frac{m^2 g}{b^2} \frac{b}{m v_0 cos\theta}}{\frac{mg}{b v_0 cos\theta} + tan\theta} \right) \frac{m v_0 cos\theta}{b}$$

$$x = \frac{m v_0 cos\theta}{b} - \frac{\frac{m^2 g}{b^2}}{\frac{mg}{b v_0 cos\theta} + tan\theta}$$

$$x = \frac{m v_0 cos\theta}{b} - \frac{\frac{m^2 g}{b^2} b v_0 cos\theta}{mg + b v_0 sin\theta}$$

$$z = \left(\frac{mg}{b v_0 cos\theta} + tan\theta \right) \left(\frac{m v_0 cos\theta}{b} - \frac{\frac{m^2 g}{b^2} b v_0 cos\theta}{mg + b v_0 sin\theta} \right) + \frac{m^2 g}{b^2} ln \left(\frac{\frac{m^2 g}{b^2} \frac{b}{m v_0 cos\theta}}{\frac{mg}{b v_0 cos\theta} + tan\theta} \right)$$

We can treat (approximately) the problem of the effect of wind on the projectile by assuming the force of air resistance to be proportional to the relative velocity of the projectile with respect to the air:

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\hat{\mathbf{z}} - b\left(\frac{d\mathbf{r}}{dt} - \mathbf{v}_w\right), \tag{3.180}$$

where $\mathbf{v}_w$ is the wind velocity. If $\mathbf{v}_w$ is constant, the term $b\mathbf{v}_w$ in Eq. (3.180) behaves as a constant force added to $-mg\hat{\mathbf{z}}$, and the problem is easily solved by the method above, the only difference being that there may be constant forces in addition to frictional forces in all three directions x, y, z.

The air resistance to a projectile decreases with altitude, so that a better form for the equation of motion of a projectile which rises to appreciable altitudes would be

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\hat{\mathbf{z}} - be^{-z/h}\frac{d\mathbf{r}}{dt}, \tag{3.181}$$

where h is the height (say about five miles) at which the air resistance falls to $1/e$ of its value at the surface of the earth. In component form,

$$\begin{gathered} m\ddot{x} = -b\dot{x}e^{-z/h}, \qquad m\ddot{y} = -b\dot{y}e^{-z/h}, \\ m\ddot{z} = -mg - b\dot{z}e^{-z/h}. \end{gathered} \tag{3.182}$$

These equations are much harder to solve.