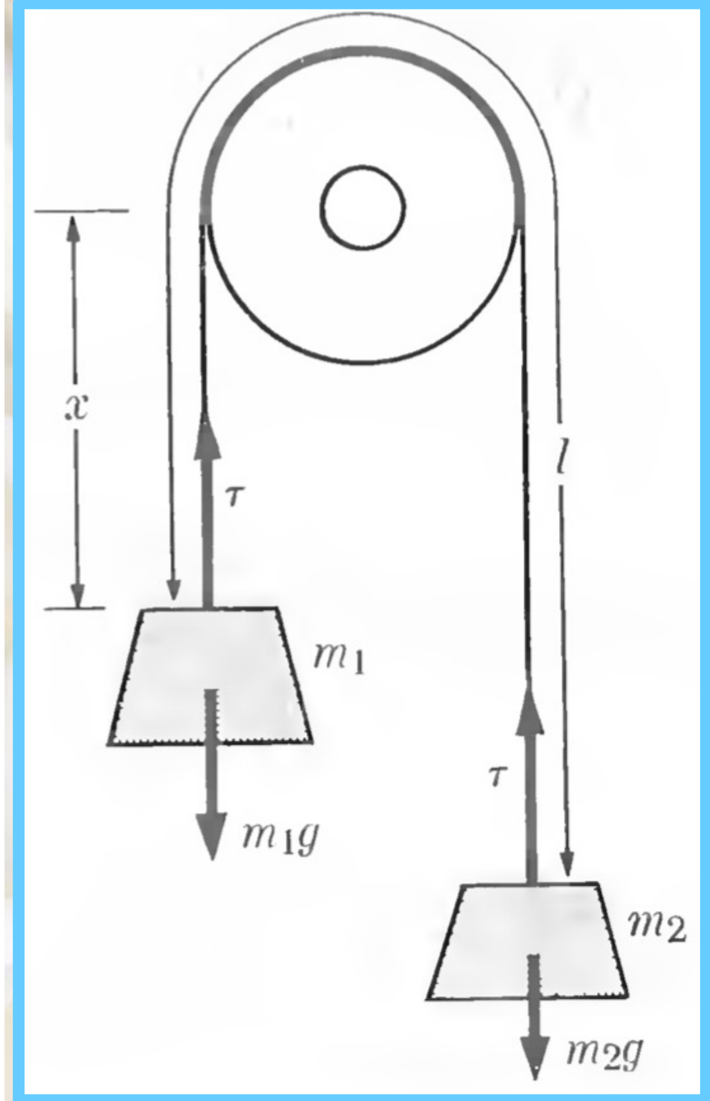
مثالهایی از دستگاه های مقید



دستگاه های مقید دستگاه هایی هستند که روی آنها یک قید یا چند قید اعمال شده باشد.

مثلا قرقره آتود نمونه یک دستگاه مقید است

در قرقره آتود طول نخ l ثابت است

ثابت بودن l نمونه یک قید است

در اینجا l را به عنوان یک مختصه تعمیم یافته در نظر می گیریم

مختصه تعمیم یافته دیگر x است

$$x_1 = x, x_2 = l - x$$

$$q_1 = x, q_2 = l$$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left(\dot{l} - \dot{x}\right)^2$$

معادلات حرکت لاگرانژ در حضور نیروهای قیدی

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k}$$

$$\xrightarrow{k=1} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1$$

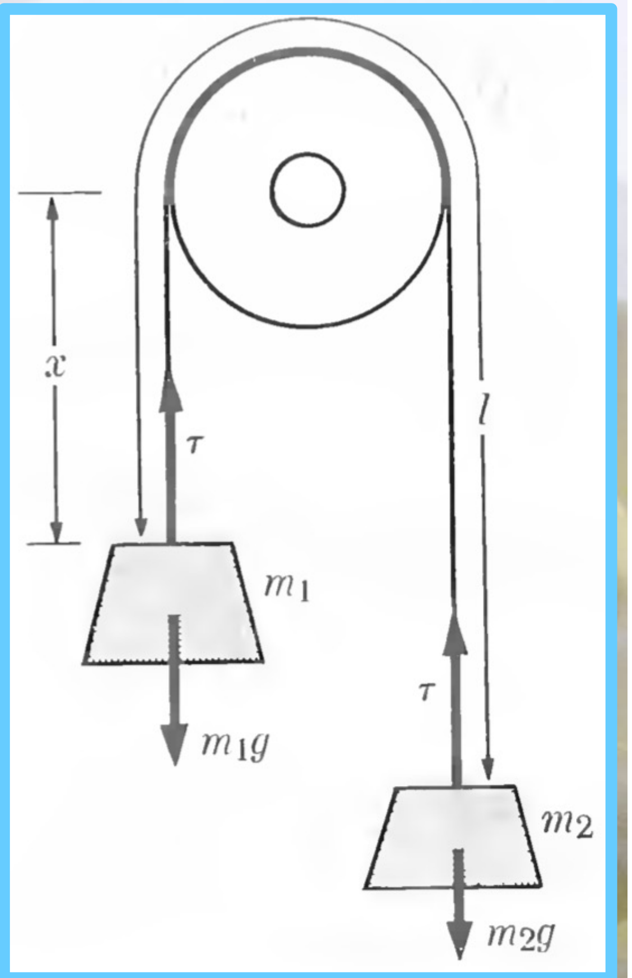
$$\mathrm{LHS}_1 \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x}$$

$$\mathrm{LHS}_1 \equiv \frac{d}{dt}\left(m_1 \dot{x} - m_2(\dot{l} - \dot{x})\right) - 0$$

$$m_1 \ddot{x} - m_2(\ddot{l} - \ddot{x}) = LHS_1$$

$$\ddot{l} = \dot{l} = 0$$

$$LHS_1 \equiv (m_1 + m_2)\ddot{x}$$



$$\xrightarrow{k=2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{l}}\right) - \frac{\partial T}{\partial l} = Q_2$$

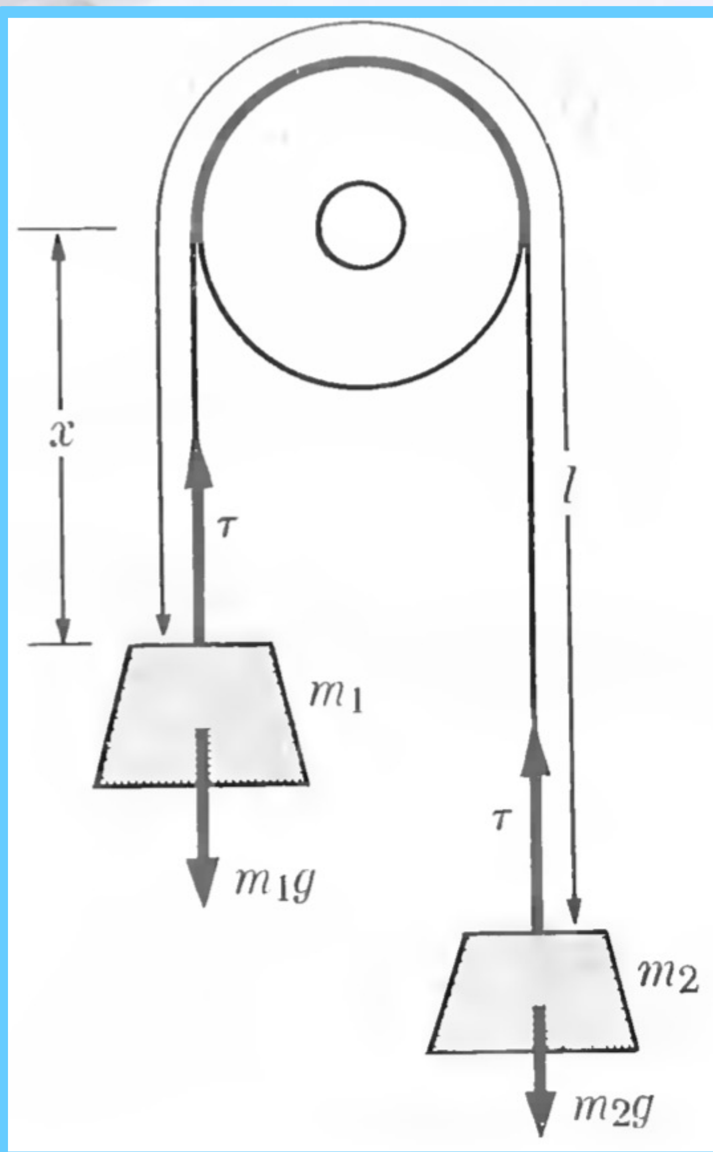
$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left(\dot{l} - \dot{x}\right)^2$$

$$\mathrm{LHS}_2 \equiv \frac{d}{dt}\left(m_2\left(\dot{l} - \dot{x}\right)\right) - 0$$

$$\mathrm{LHS}_2 \equiv m_2\left(\ddot{l} - \ddot{x}\right)$$

$$\ddot{l} = \dot{l} = 0$$

$$\mathrm{LHS}_2 \equiv -m_2 \ddot{x}$$



$$\delta W_k = Q_k \delta q_k$$

$$\xrightarrow{k=1} \xRightarrow{q_1=x} \delta W_1 = Q_1 \delta x$$

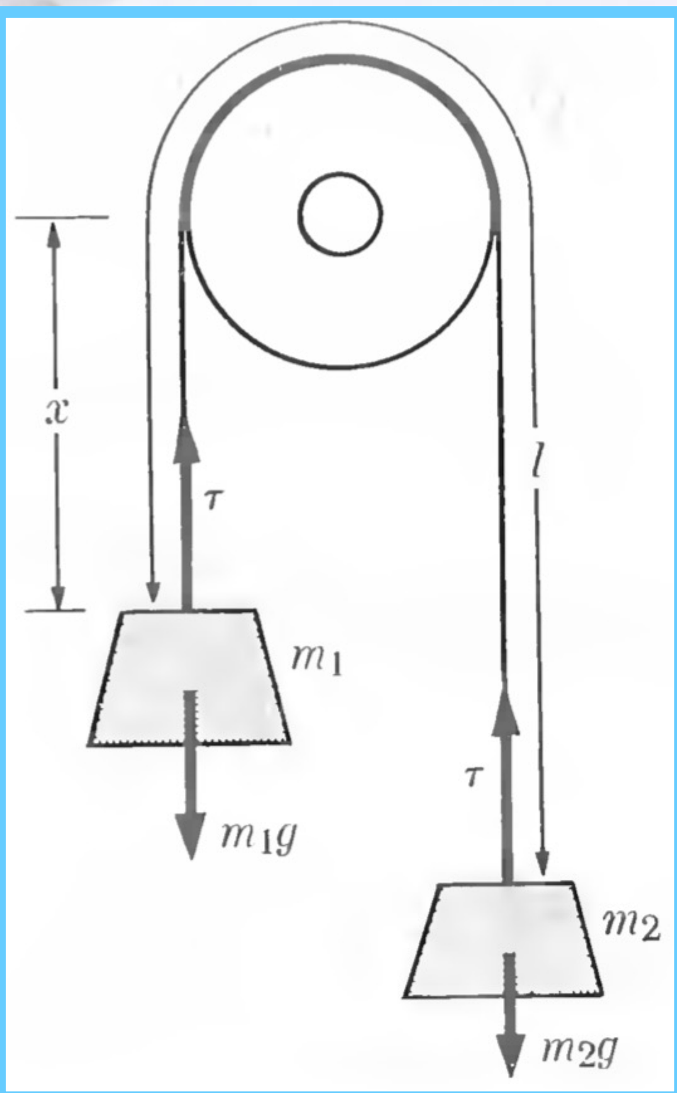
$$Q_k = \sum_{i=1}^{N} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k} \right)$$

$$Q_k = \sum_{i=1}^{2} \left(F_{ix} \frac{\partial x_i}{\partial q_k} \right) = F_{1x} \frac{\partial x_1}{\partial q_k} + F_{2x} \frac{\partial x_2}{\partial q_k}$$

$$\xrightarrow{k=1} \xRightarrow{q_1=x} Q_1 = F_{1x} \frac{\partial x}{\partial x} + F_{2x} \frac{\partial (l-x)}{\partial x}$$

$$Q_1 = F_{1x} - F_{2x} = (m_1 g - \tau) - (m_2 g - \tau)$$

$$\mathrm{RHS}_1 \equiv Q_1 \equiv Q_x = (m_1 - m_2) g$$



$$\delta W_k = Q_k \delta q_k$$

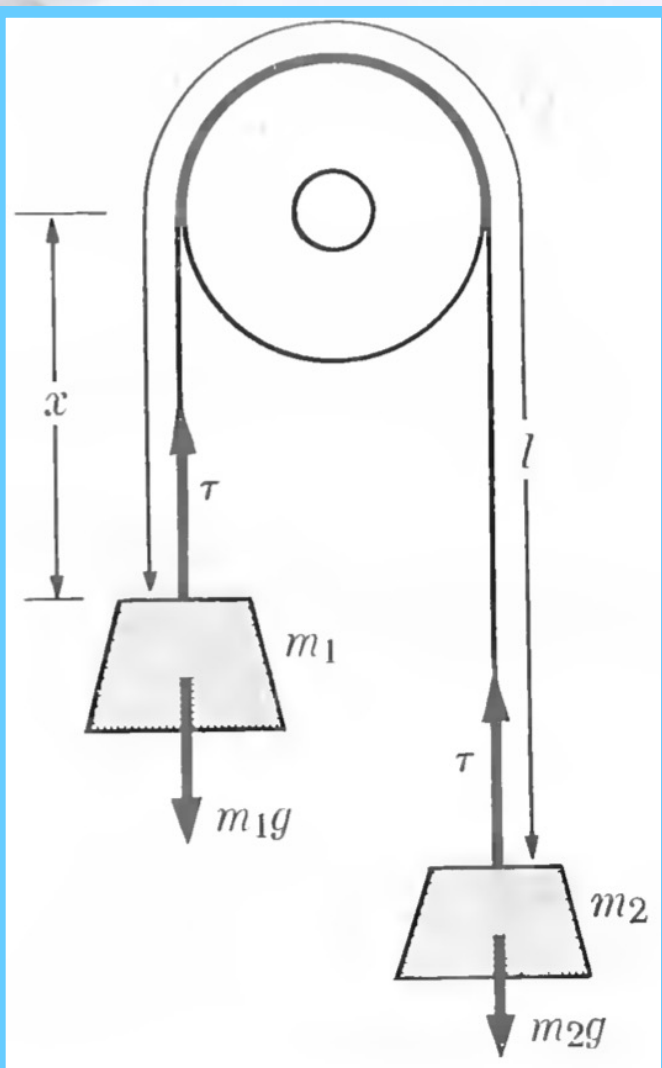
$$\xrightarrow{k=2} \xRightarrow{q_2=l} \delta W_2 = Q_2 \delta l$$

$$Q_k = \sum_{i=1}^{2} \left(F_{ix} \frac{\partial x_i}{\partial q_k} \right) = F_{1x} \frac{\partial x_1}{\partial q_k} + F_{2x} \frac{\partial x_2}{\partial q_k}$$

$$\xrightarrow{k=2} \xRightarrow{q_2=l} Q_2 = F_{1x} \frac{\partial x}{\partial l} + F_{2x} \frac{\partial (l-x)}{\partial l}$$

$$Q_2 = (m_1 g - \tau)\times 0 + (m_2 g - \tau)\times 1$$

$$\mathrm{RHS}_2 \equiv Q_2 \equiv Q_l = m_2 g - \tau$$



$$\mathrm{LHS}_1 \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = (m_1 + m_2)\ddot{x}$$

$$\mathrm{LHS}_2 \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{l}}\right) - \frac{\partial T}{\partial l} = -m_2\ddot{x}$$

$$\mathrm{RHS}_1 \equiv Q_1 \equiv Q_x = (m_1 - m_2)g$$

$$\mathrm{RHS}_2 \equiv Q_2 \equiv Q_l = m_2 g - \tau$$

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

$$-m_2\ddot{x} = m_2 g - \tau$$

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

$$-m_2 \frac{m_1 - m_2}{m_1 + m_2} g = m_2 g - \tau$$

$$\tau = m_2\left(1 + \frac{m_1 - m_2}{m_1 + m_2}\right) g$$

$$\tau = \frac{2 m_1 m_2}{m_1 + m_2} g$$

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

$$\frac{dv_x}{dt} = \frac{m_1 - m_2}{m_1 + m_2} g$$

$$dv_x = \frac{m_1 - m_2}{m_1 + m_2} g dt$$

$$\int_{v_0}^{v} dv_x = \int_0^t \frac{m_1 - m_2}{m_1 + m_2} g dt$$

$$v - v_0 = \frac{m_1 - m_2}{m_1 + m_2} g t$$

$$v = v_0 + \frac{m_1 - m_2}{m_1 + m_2} g t$$

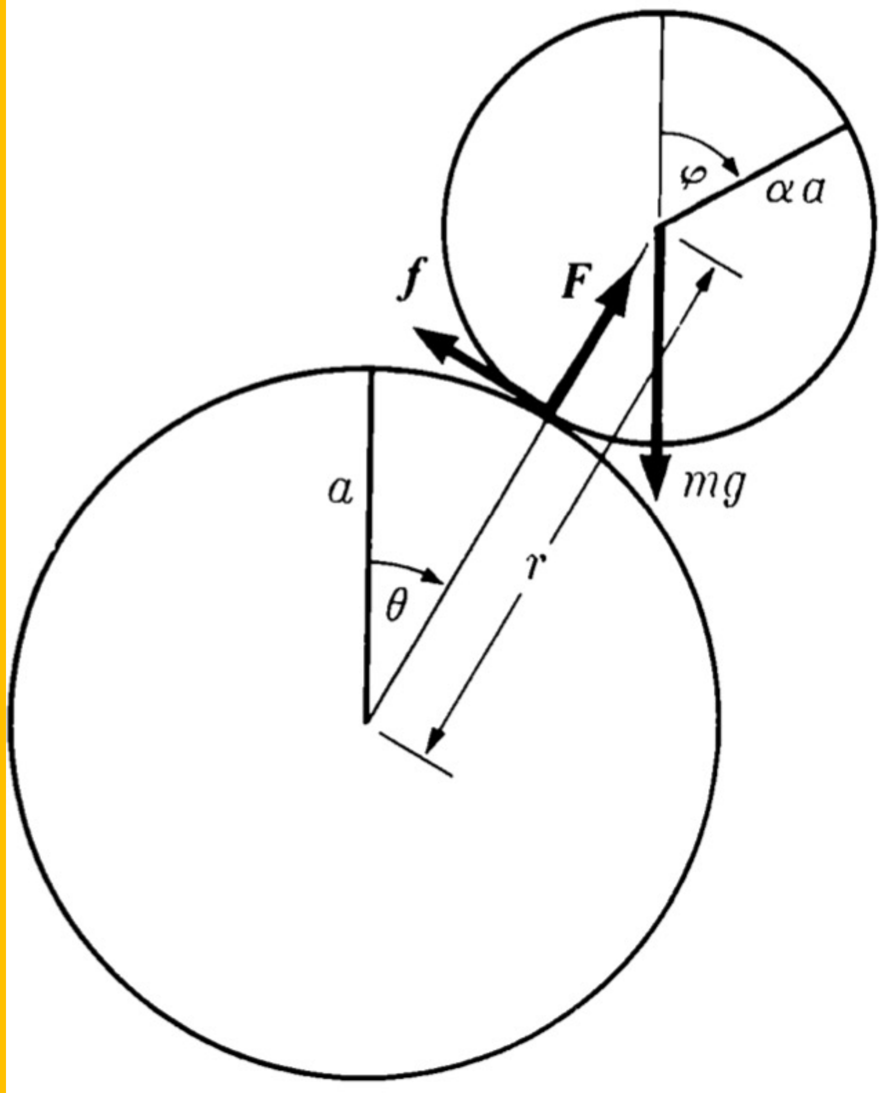
$$\frac{dx}{dt} = v_0 + \frac{m_1 - m_2}{m_1 + m_2} g t$$

$$\int_{x_0}^{x} dx = \int_0^t \left(v_0 + \frac{m_1 - m_2}{m_1 + m_2} g t \right) dt$$

$$x - x_0 = v_0 t + \frac{1}{2} \frac{m_1 - m_2}{m_1 + m_2} g t^2$$

$$x = x_0 + v_0 t + \frac{1}{2} \frac{m_1 - m_2}{m_1 + m_2} g t^2$$

مثال: استوانه‌ای روی استوانه دیگر با ضریب اصطحکاک غیر صفر مطابق شکل قرار داده شده است.



الف) زاویه‌ای که استوانه بالایی از غلتش به لغزش روی می‌آورد را تعیین کنید

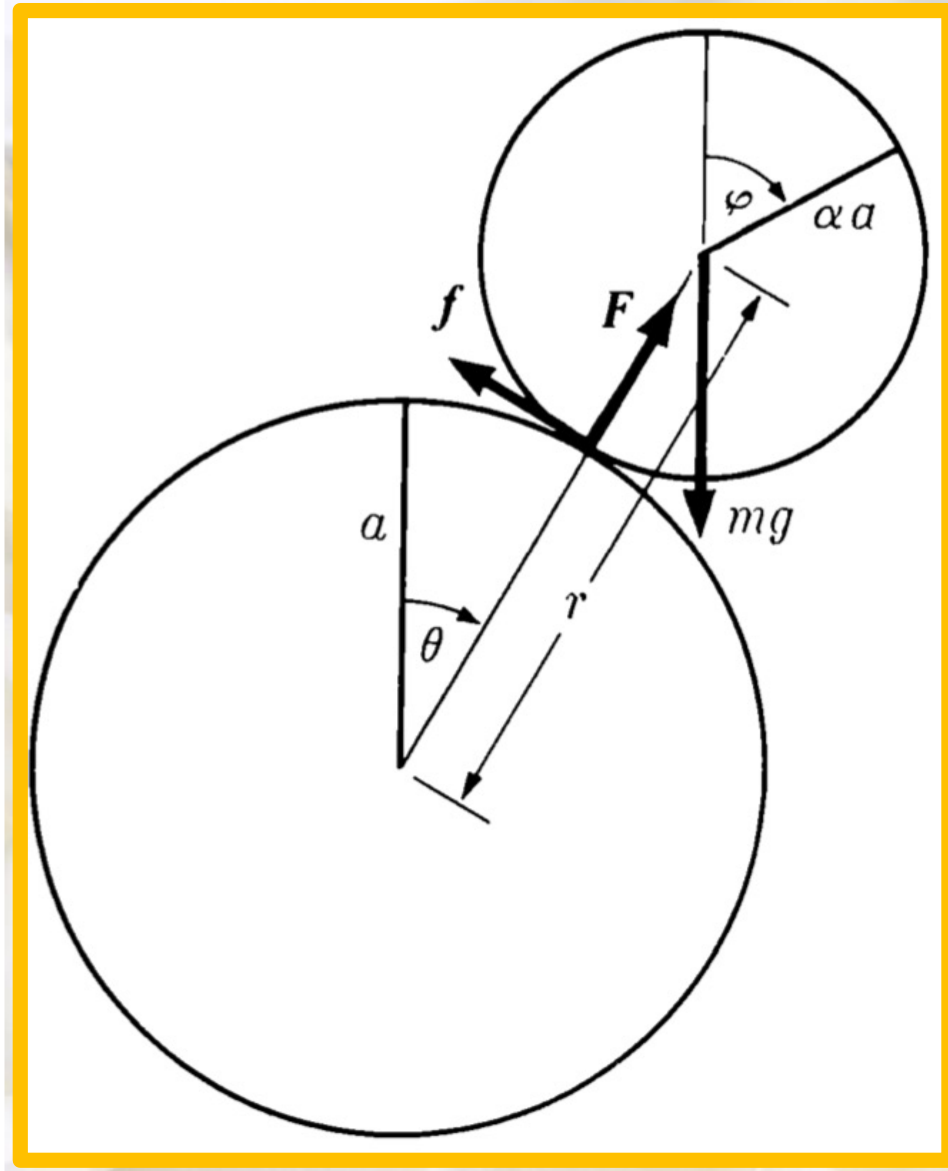
ب) زاویه‌ای که استوانه بالایی از لغزش به جدایی از سطح استوانه زیرین روی می‌آورد را تعیین کنید

شرط غلتش $f = \mu F$

شرط یا شروع لغزش $f \le \mu F$

شرط جدایی از سطح $F = 0$

مساله را به سه بخش تقسیم می کنیم



الف) بخش اول: غلتش

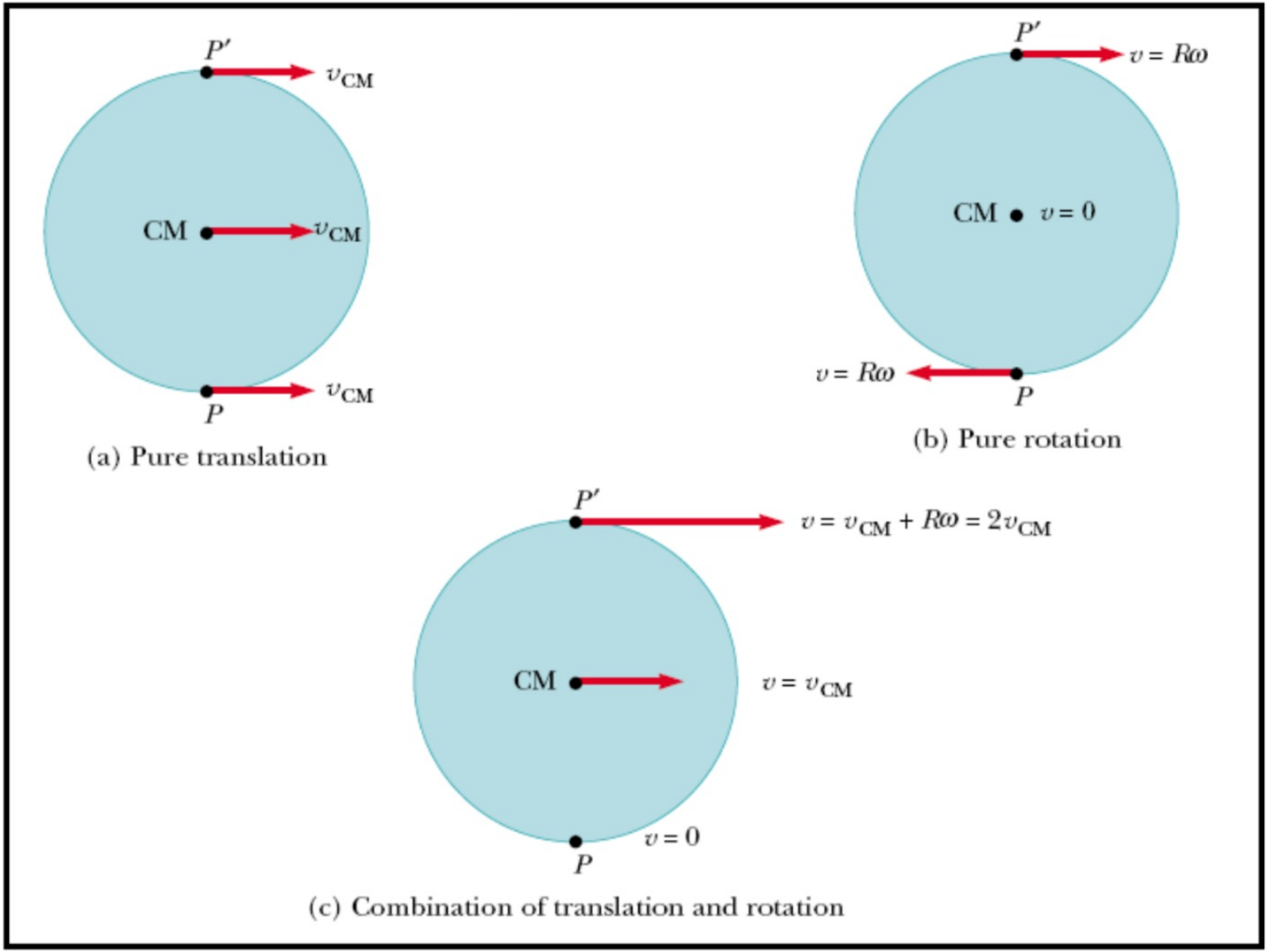
ب) بخش دوم: لغزش

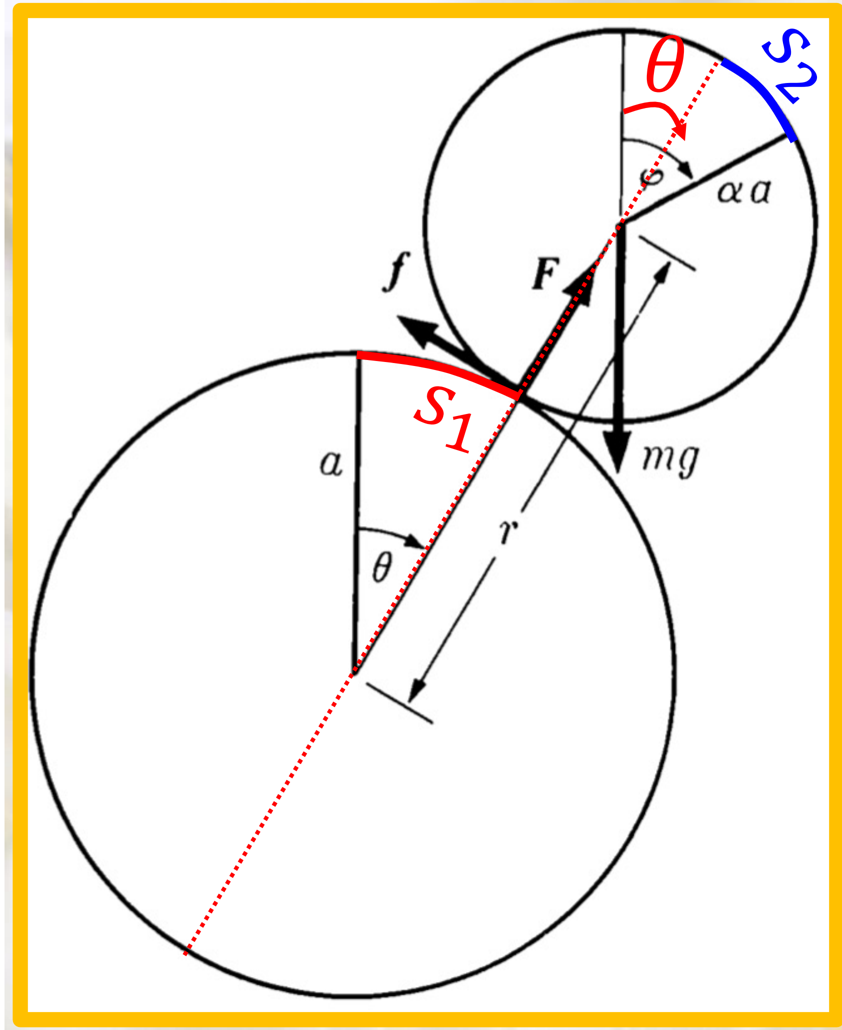
ج) بخش سوم: جدایی

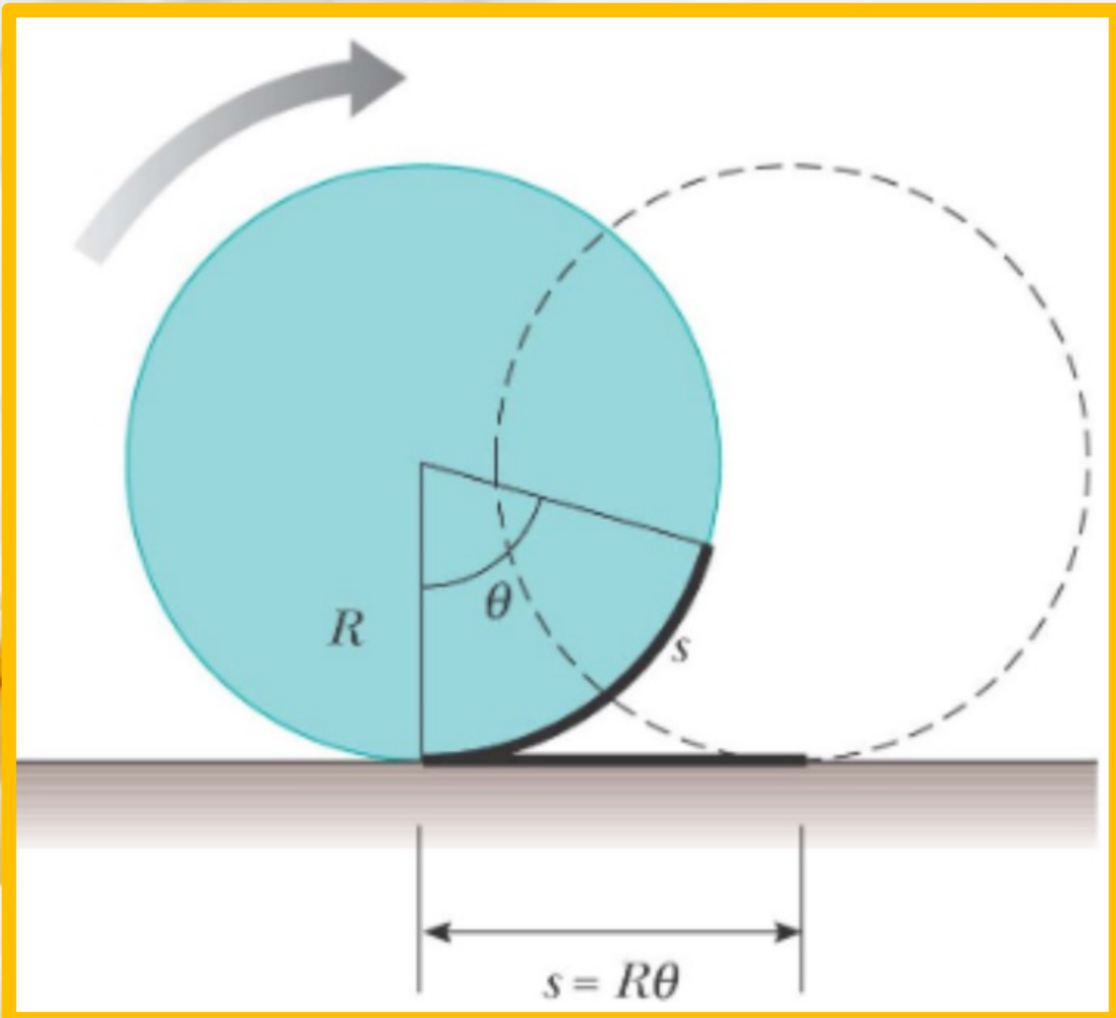
تا وقتی که دو استوانه از هم جدا نشده‌اند، یعنی تا زمانی که نیروی عمود بر سطح F غیر صفر است فاصله بین دو استوانه r ثابت باقی می‌ماند و لذا قید زیر را داریم:

$$r = a + \alpha a = cte.$$

$$r = (1 + \alpha)a = cte.$$


P′
v_{CM}
CM
v_{CM}
v_{CM}
P
(a) Pure translation
P′
$v = R\omega$
CM
$v = 0$
$v = R\omega$
P
(b) Pure rotation
P′
$v = v_{CM} + R\omega = 2v_{CM}$
CM
$v = v_{CM}$
$v = 0$
P
(c) Combination of translation and rotation





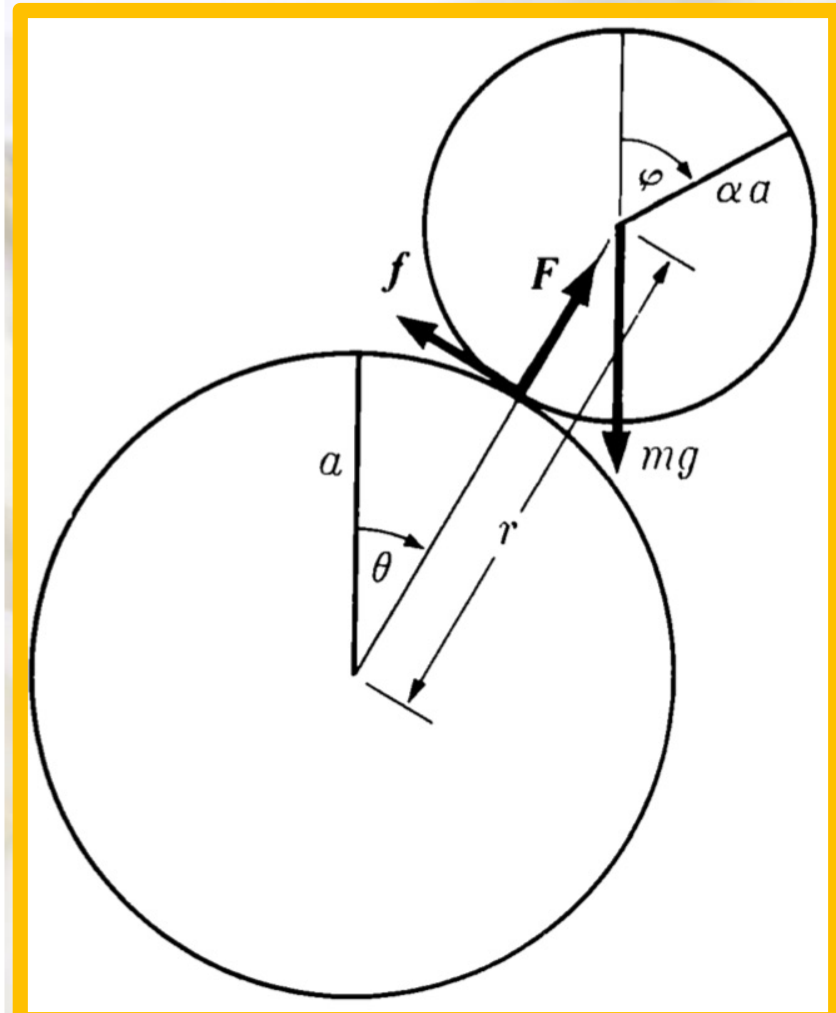
الف) اگر غلتش خالص روی دهد، معادله قیدی زیر را خواهیم داشت:

$$s_1 \equiv a\theta = \alpha a(\varphi - \theta) \equiv s_2$$

$$a\theta + \alpha a\theta = \alpha a\varphi$$

$$(1+\alpha)\theta = \alpha\varphi$$

ب) زاویه لغزش γ حول استوانه ثابت به صورت زیر تعریف می‌شود:



$$(1+\alpha)\theta = \alpha\varphi$$

$$\theta = \frac{\alpha}{1+\alpha}\varphi$$

غلتش

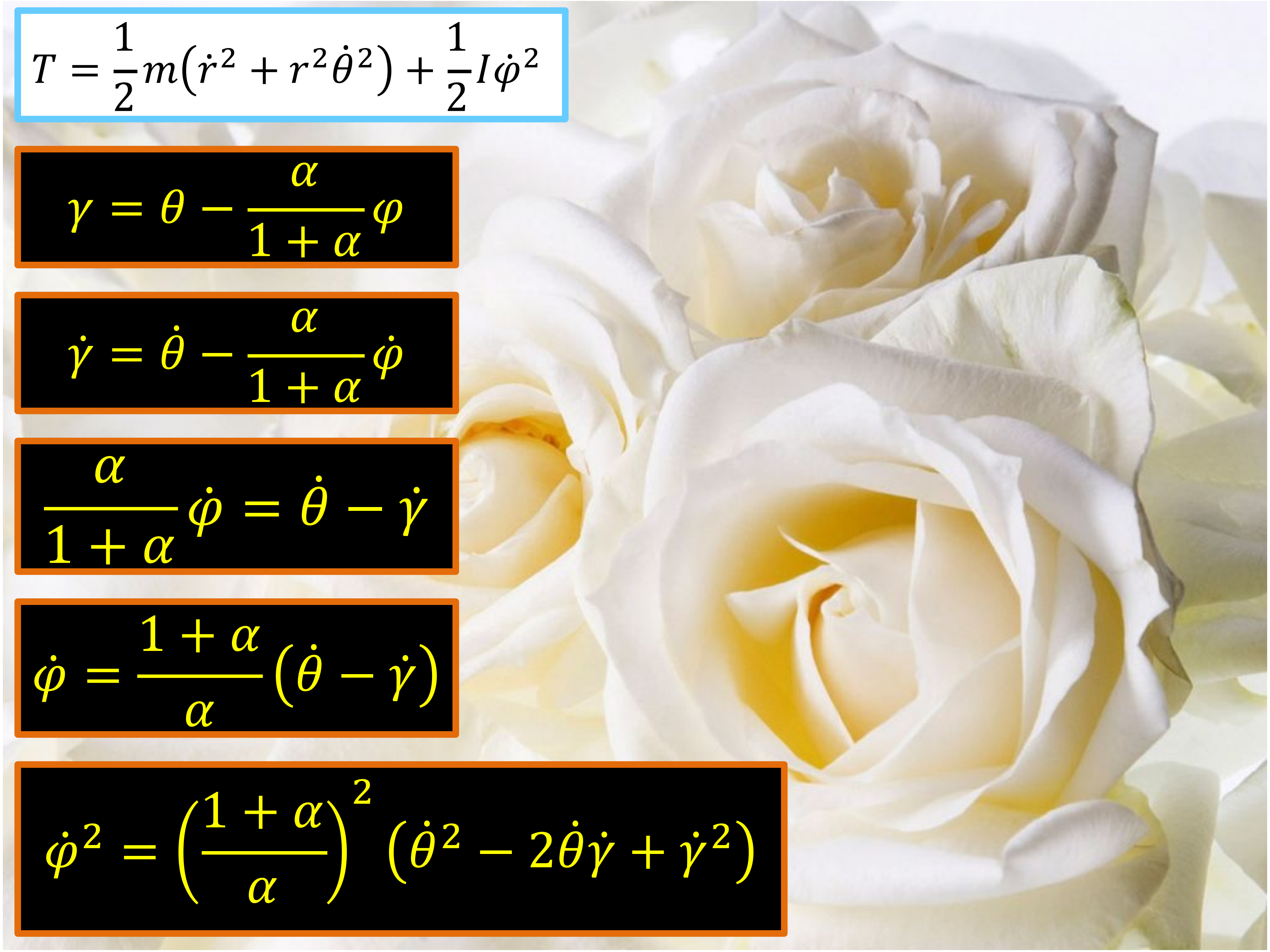
$$0 = \theta - \frac{\alpha}{1+\alpha}\varphi$$

لغزش

$$\gamma = \theta - \frac{\alpha}{1+\alpha}\varphi$$

یعنی تا وقتی که استوانه بدون سرخودرن بغلتد

$$\gamma = 0$$



$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}I\dot{\varphi}^2$$

$$\gamma = \theta - \frac{\alpha}{1+\alpha}\varphi$$

$$\dot{\gamma} = \dot{\theta} - \frac{\alpha}{1+\alpha}\dot{\varphi}$$

$$\frac{\alpha}{1+\alpha}\dot{\varphi} = \dot{\theta} - \dot{\gamma}$$

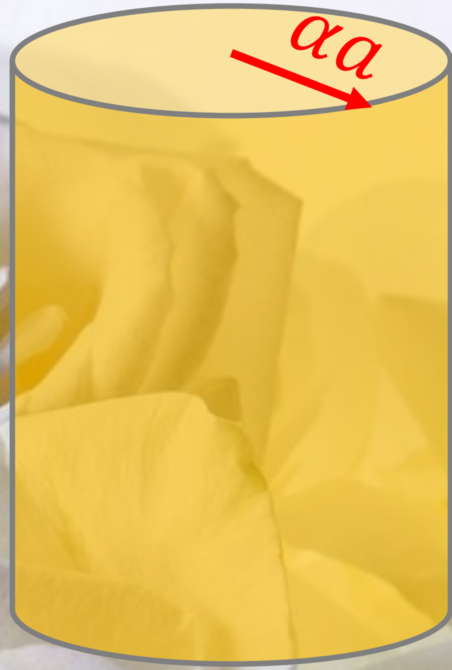
$$\dot{\varphi} = \frac{1+\alpha}{\alpha}\left(\dot{\theta} - \dot{\gamma}\right)$$

$$\dot{\varphi}^2 = \left(\frac{1+\alpha}{\alpha}\right)^2\left(\dot{\theta}^2 - 2\dot{\theta}\dot{\gamma} + \dot{\gamma}^2\right)$$

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}I\dot{\varphi}^2$$

$$\dot{\varphi}^2 = \left(\frac{1+\alpha}{\alpha}\right)^2\left(\dot{\theta}^2 - 2\dot{\theta}\dot{\gamma} + \dot{\gamma}^2\right)$$

$$I = \int r^2 dm = \int r^2 \rho dV = \rho\int_{r=0}^{\alpha a}\int_{\theta=0}^{2\pi}\int_{z=0}^{h} r^2(rdrd\theta dz)$$



$$I = \rho\frac{1}{4}(\alpha a)^4\times 2\pi h = \frac{1}{2}(\alpha a)^2\{\rho[\pi(\alpha a)^2 h]\} = \frac{1}{2}(\alpha a)^2\{\rho V\} = \frac{1}{2}(\alpha a)^2\{m\} = \frac{1}{2}m(\alpha a)^2$$

$$I = \frac{1}{2}m(\alpha a)^2 = \frac{1}{2}m\alpha^2 a^2$$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}\times\frac{1}{2}m\alpha^2 a^2\left(\frac{1+\alpha}{\alpha}\right)^2\left(\dot{\theta}^2 - 2\dot{\theta}\dot{\gamma} + \dot{\gamma}^2\right)$$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{4}ma^2(1+\alpha)^2\left(\dot{\theta}^2 - 2\dot{\theta}\dot{\gamma} + \dot{\gamma}^2\right)$$

$$T=\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\dot{\theta}^2+\frac{1}{4}ma^2(1+\alpha)^2(\dot{\theta}^2-2\dot{\theta}\dot{\gamma}+\dot{\gamma}^2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)=Q_k+\frac{\partial T}{\partial q_k}$$

$$\xrightarrow{k=1}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right)-\frac{\partial T}{\partial q_1}=Q_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right)-\frac{\partial T}{\partial r}=Q_1\equiv Q_r$$

$$\mathrm{LHS}_1=\frac{d}{dt}(m\dot{r})-mr\dot{\theta}^2=m\ddot{r}-mr\dot{\theta}^2$$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{4}ma^2(1+\alpha)^2\left(\dot{\theta}^2 - 2\dot{\theta}\dot{\gamma} + \dot{\gamma}^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k}$$

$$\xrightarrow{k=2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_2 \equiv Q_\theta$$

$$\mathrm{LHS}_2 = \frac{d}{dt}\left(mr^2\dot{\theta} + \frac{1}{4}ma^2(1+\alpha)^2\left(2\dot{\theta} - 2\dot{\gamma}\right)\right) - 0$$

$$\mathrm{LHS}_2 = \frac{d}{dt}\left(mr^2\dot{\theta} + \frac{1}{2}ma^2(1+\alpha)^2(\dot{\theta} - \dot{\gamma})\right) - 0$$

$$\mathrm{LHS}_2 = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + \frac{1}{2}ma^2(1+\alpha)^2(\ddot{\theta} - \ddot{\gamma})$$

$$\mathrm{LHS}_2 = 2mr\dot{r}\dot{\theta} + m\left(r^2 + \frac{1}{2}a^2(1+\alpha)^2\right)\ddot{\theta} - \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma}$$

$$\mathrm{LHS}_2 = m\left(r^2 + \frac{1}{2}a^2(1+\alpha)^2\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} - \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma}$$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{4}ma^2(1+\alpha)^2(\dot{\theta}^2 - 2\dot{\theta}\dot{\gamma} + \dot{\gamma}^2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k}$$

$$\xrightarrow{k=3} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) - \frac{\partial T}{\partial q_3} = Q_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\gamma}}\right) - \frac{\partial T}{\partial \gamma} = Q_3 \equiv Q_\gamma$$

$$\mathrm{LHS}_3 = \frac{d}{dt}\left(\frac{1}{4}ma^2(1+\alpha)^2(-2\dot{\theta} + 2\dot{\gamma})\right) - 0$$

$$\mathrm{LHS}_3 = \frac{d}{dt}\left(\frac{1}{2}ma^2(1+\alpha)^2(-\dot{\theta}+\dot{\gamma})\right)$$

$$\mathrm{LHS}_3 = \frac{1}{2}ma^2(1+\alpha)^2(-\ddot{\theta}+\ddot{\gamma})$$

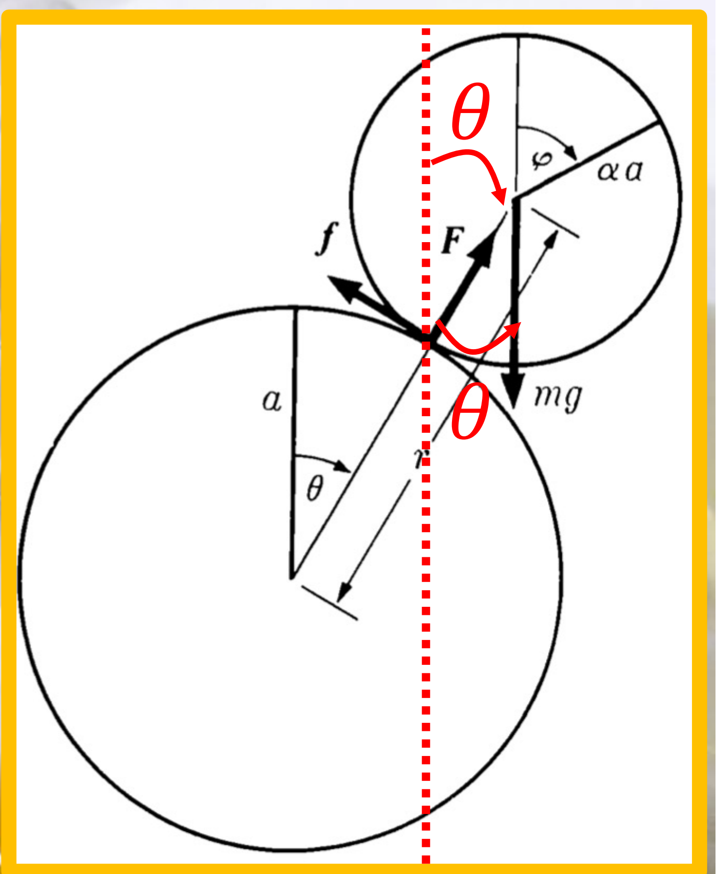
$$\mathrm{LHS}_3 = -\frac{1}{2}ma^2(1+\alpha)^2\ddot{\theta} + \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma}$$

$$\delta W_k = Q_k \delta q_k$$

$$\xrightarrow{k=1} \xRightarrow{q_1 = r} \delta W_1 = Q_1 \delta r$$

$$Q_k = \sum_{i=1}^{N} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k} \right)$$

$$Q_k = \sum_{i=1}^{1} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} \right) = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k}$$



$$x = r \sin\theta$$

$$y = r \cos\theta$$

$$\xrightarrow{k=1} \xRightarrow{q_1 = r} Q_1 = F_x \frac{\partial x}{\partial r} + F_y \frac{\partial y}{\partial r}$$

$$\frac{\partial x}{\partial r} = \sin\theta$$

$$\frac{\partial y}{\partial r} = \cos\theta$$

$$\mathrm{RHS}_1 \equiv Q_1 \equiv Q_r = (F \sin\theta)\sin\theta + (F\cos\theta - mg)\cos\theta$$

$$\mathrm{RHS}_1 \equiv Q_1 \equiv Q_r = F\left(\sin^2\theta + \cos^2\theta\right) - mg\cos\theta$$

$$\mathrm{RHS}_1 \equiv Q_1 \equiv Q_r = F - mg\cos\theta$$

$$\delta W_k = Q_k \delta q_k$$

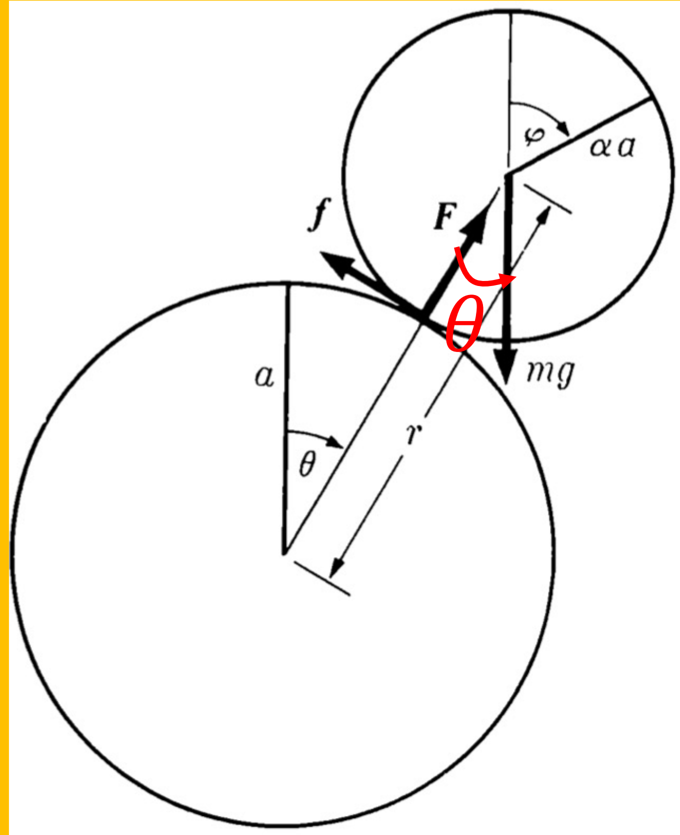
$$\xrightarrow{k=2} \xRightarrow{q_2=\theta} \delta W_2 = Q_2 \delta\theta$$

$$Q_k = \sum_{i=1}^{N} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k} \right)$$

$$Q_k = \sum_{i=1}^{1} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} \right) = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k}$$

$$x = r \sin\theta$$

$$y = r \cos\theta$$



$$\xrightarrow{k=2} \xRightarrow{q_2=\theta} Q_2 = F_x \frac{\partial x}{\partial\theta} + F_y \frac{\partial y}{\partial\theta}$$

$$\frac{\partial x}{\partial\theta} = r\cos\theta$$

$$\frac{\partial y}{\partial\theta} = -\mathrm{r}\sin\theta$$

$$\mathrm{RHS}_2 \equiv Q_2 \equiv Q_\theta = (-mg)\,(-\mathrm{r}\sin\theta)$$

$$\mathrm{RHS}_2 \equiv Q_2 \equiv Q_\theta = mgr\sin\theta$$

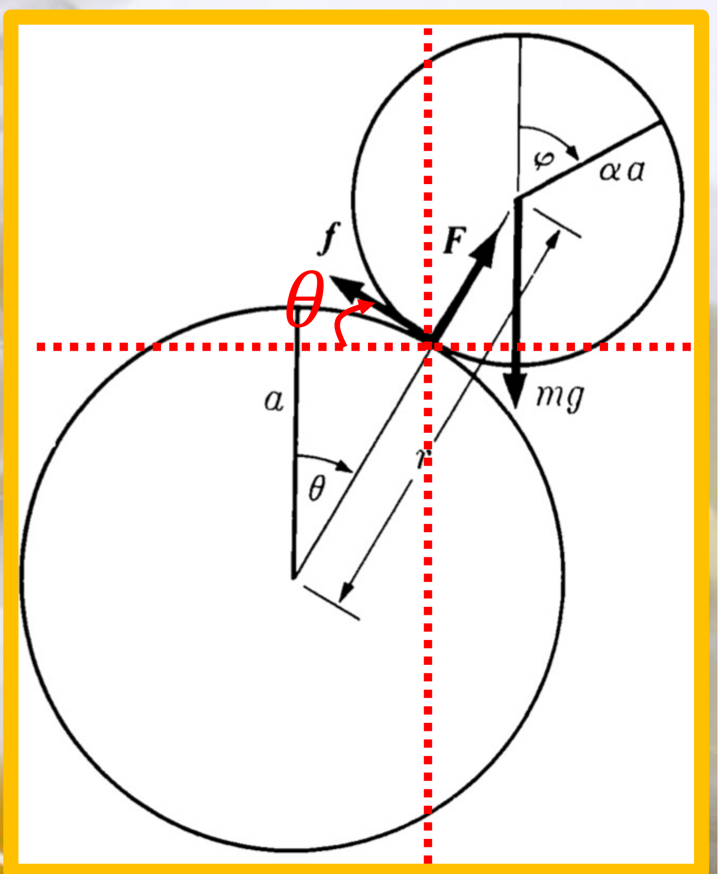
$$\delta W_k = Q_k \delta q_k$$

$$\xrightarrow{k=3} \xRightarrow{q_3=\gamma} \delta W_3 = Q_3 \delta\gamma$$

$$Q_k = \sum_{i=1}^{N} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k} \right)$$

$$Q_k = \sum_{i=1}^{1} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} \right) = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k}$$

$$x = r \sin\theta \qquad y = r\cos\theta \qquad \gamma = \theta - \frac{\alpha}{1+\alpha}\varphi$$



$$\xrightarrow{k=3} \xRightarrow{q_3=\gamma} Q_3 = F_x \frac{\partial x}{\partial\gamma} + F_y \frac{\partial y}{\partial\gamma} \qquad \frac{\partial x}{\partial\gamma} = r\cos\theta \qquad \frac{\partial y}{\partial\gamma} = -\mathrm{r}\sin\theta$$

$$\mathrm{RHS}_3 \equiv Q_3 \equiv Q_\gamma = (-f\cos\theta) r\cos\theta + (f\sin\theta)(-r\sin\theta)$$

$$\mathrm{RHS}_3 \equiv Q_3 \equiv Q_\gamma = -rf(\cos^2\theta + \sin^2\theta) = -fr$$

$$x = r\sin\theta \qquad y = r\cos\theta \qquad \gamma = \theta - \frac{\alpha}{1+\alpha}\varphi$$

$$x = r\sin\left(\gamma + \frac{\alpha}{1+\alpha}\varphi\right) \qquad y = r\cos\left(\gamma + \frac{\alpha}{1+\alpha}\varphi\right)$$

$$\left.\frac{\partial x}{\partial \gamma}\right)_{r,\theta} = \left.\frac{\partial x}{\partial \varphi}\right)_{r,\theta} \left.\frac{\partial \varphi}{\partial \gamma}\right)_{r,\theta} = -r\,\frac{\alpha}{1+\alpha}\,\frac{1+\alpha}{\alpha}\cos\left(\gamma + \frac{\alpha}{1+\alpha}\varphi\right) = -r\cos\theta$$

$$\left.\frac{\partial y}{\partial \gamma}\right)_{r,\theta} = \left.\frac{\partial y}{\partial \varphi}\right)_{r,\theta} \left.\frac{\partial \varphi}{\partial \gamma}\right)_{r,\theta} = r\,\frac{\alpha}{1+\alpha}\,\frac{1+\alpha}{\alpha}\sin\left(\gamma + \frac{\alpha}{1+\alpha}\varphi\right) = r\sin\theta$$

$$\delta W = \sum_{k=1}^{3N} Q_k \, \delta q_k , \tag{9.28}$$

where

$$Q_k = \sum_{i=1}^{N} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k} \right). \tag{9.29}$$

The coefficients Q_k depend on the forces acting on the particles, on the coordinates $q_1, \ldots, q_{3N}$, and possibly also on the time t. In view of the similarity in form between Eqs. (9.26) and (9.28), it is natural to call the quantity Q_k the *generalized force* associated with the coordinate q_k. We can define the generalized force Q_k directly, without reference to the cartesian coordinate system, as the coefficient which determines the work done in a virtual displacement in which q_k alone changes:

$$\delta W_k = Q_k \, \delta q_k , \tag{9.30}$$

where δW_k is the work done when the system moves in such a way that q_k increases by δq_k, all other coordinates remaining constant. Notice that the work in Eq. (9.26), and therefore also in Eq. (9.30), is to be computed from the values of the forces for the positions $x_1, \ldots, z_N$, or $q_1, \ldots, q_{3N}$; that is, we do not take account of any change in the forces during the virtual displacement.

acting to increase θ. It is usually quicker to use the definition (9.30), which enables us to bypass the cartesian coordinates altogether. If we consider a small dis-

$$\delta W_k = Q_k \delta q_k$$

$$\xrightarrow{k=1}\xRightarrow{q_1=r} \delta W_1 = Q_1 \delta r$$

$$\delta W_1 = Q_1 \delta r \equiv (\delta W_r = Q_r \delta r) = (F - mg\cos\theta)\delta r$$

$$\mathrm{RHS}_1 \equiv Q_1 \equiv Q_r = F - mg\cos\theta$$

$$\xrightarrow{k=2}\xRightarrow{q_2=\theta} \delta W_2 = Q_2 \delta\theta$$

$$\delta W_2 = Q_2 \delta\theta \equiv (\delta W_\theta = Q_\theta \delta\theta) = mgr\sin\theta\,\delta\theta$$

$$\mathrm{RHS}_2 \equiv Q_2 \equiv Q_\theta = mgr\sin\theta$$

$$\xrightarrow{k=3}\xRightarrow{q_3=\gamma} \delta W_3 = Q_3 \delta\gamma$$

$$\delta W_3 = Q_3 \delta\gamma \equiv \left(\delta W_\gamma = Q_\gamma \delta\gamma\right) = -fr\delta\gamma$$

$$\mathrm{RHS}_3 \equiv Q_3 \equiv Q_\gamma = -fr$$

$$\mathrm{LHS}_1 = m\ddot{r} - mr\dot{\theta}^2$$

$$\mathrm{LHS}_2 = m\left(r^2 + \frac{1}{2}a^2(1+\alpha)^2\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} - \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma}$$

$$\mathrm{LHS}_3 = -\frac{1}{2}ma^2(1+\alpha)^2\ddot{\theta} + \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma}$$

$$\mathrm{RHS}_1 \equiv Q_1 \equiv Q_r = F - mg\cos\theta$$

$$\mathrm{RHS}_2 \equiv Q_2 \equiv Q_\theta = mgr\sin\theta$$

$$\mathrm{RHS}_3 \equiv Q_3 \equiv Q_\gamma = -fr$$

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 = F - mg\cos\theta \\ m\left(r^2 + \frac{1}{2}a^2(1+\alpha)^2\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} - \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma} = mgr\sin\theta \\ -\frac{1}{2}ma^2(1+\alpha)^2\ddot{\theta} + \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma} = -fr \end{cases}$$

دسته معادلات لاگرانژ

حال می توانیم قیود مساله را بر دسته معادلات لاگرانژ اعمال کنیم

قید غلتش

$$\gamma = 0$$

قید بودن روی سطح استوانه

$$r = (1+\alpha)a = cte.$$

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 = F - mg\cos\theta \\ m\left(r^2 + \frac{1}{2}a^2(1+\alpha)^2\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} - \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma} = mgr\sin\theta \\ -\frac{1}{2}ma^2(1+\alpha)^2\ddot{\theta} + \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma} = -fr \end{cases}$$

$$\begin{cases} 0 - m(1+\alpha)a\dot{\theta}^2 = F - mg\cos\theta \\ m\left(a^2(1+\alpha)^2 + \frac{1}{2}a^2(1+\alpha)^2\right)\ddot{\theta} + 0 - 0 = mg(1+\alpha)a\sin\theta \\ -\frac{1}{2}ma^2(1+\alpha)^2\ddot{\theta} + 0 = -f(1+\alpha)a \end{cases}$$

$$\begin{cases} -m(1+\alpha)a\dot{\theta}^2 = F - mg\cos\theta \\ \frac{3}{2}ma^2(1+\alpha)^2\ddot{\theta} = mg(1+\alpha)a\sin\theta \\ -\frac{1}{2}ma^2(1+\alpha)^2\ddot{\theta} = -f(1+\alpha)a \end{cases}$$

دسته معادلات ساده شده لاگرانژ پس از اعمال قیود غلتش و باقی ماندن استوانه بالایی روی سطح استوانه زیرین

$$\begin{cases} F = mg\cos\theta - (1+\alpha)ma\dot{\theta}^2 \\ \frac{3}{2}m(1+\alpha)^2a^2\ddot{\theta} = (1+\alpha)mga\sin\theta \\ f = \frac{1}{2}(1+\alpha)ma\ddot{\theta} \end{cases}$$

معادله دوم (وسطی) لاگرانژ عمدا ساده تر نشده است تا بتوان راحتر با روش انرژی آن را حل کرد

$$\frac{3}{2}m(1+\alpha)^2a^2\ddot{\theta} = (1+\alpha)mga\sin\theta$$

حل معادله معادله دوم لاگرانژ به روش انرژی

ابتدا انرژی جنبشی زیر را با اعمال قیود ساده می کنیم

$$T=\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\dot{\theta}^2+\frac{1}{4}ma^2(1+\alpha)^2(\dot{\theta}^2-2\dot{\theta}\dot{\gamma}+\dot{\gamma}^2)$$

$$\gamma = 0$$

$$r=(1+\alpha)a=cte.$$

$$T=0+\frac{1}{2}m(1+\alpha)^2a^2\dot{\theta}^2+\frac{1}{4}ma^2(1+\alpha)^2(\dot{\theta}^2-0+0)$$

$$T=\frac{3}{4}(1+\alpha)^2ma^2\dot{\theta}^2$$

حال انرژی پتانسیل را می‌نویسیم:

$$V = mgr\cos\theta$$

$$V = mga(1+\alpha)\cos\theta$$

سپس با در نظر گرفتن انرژی جنبشی به دست آمده زیر:

$$T = \frac{3}{4}(1+\alpha)^2 ma^2\dot{\theta}^2$$

انرژی کل را به صورت زیر می نویسیم:

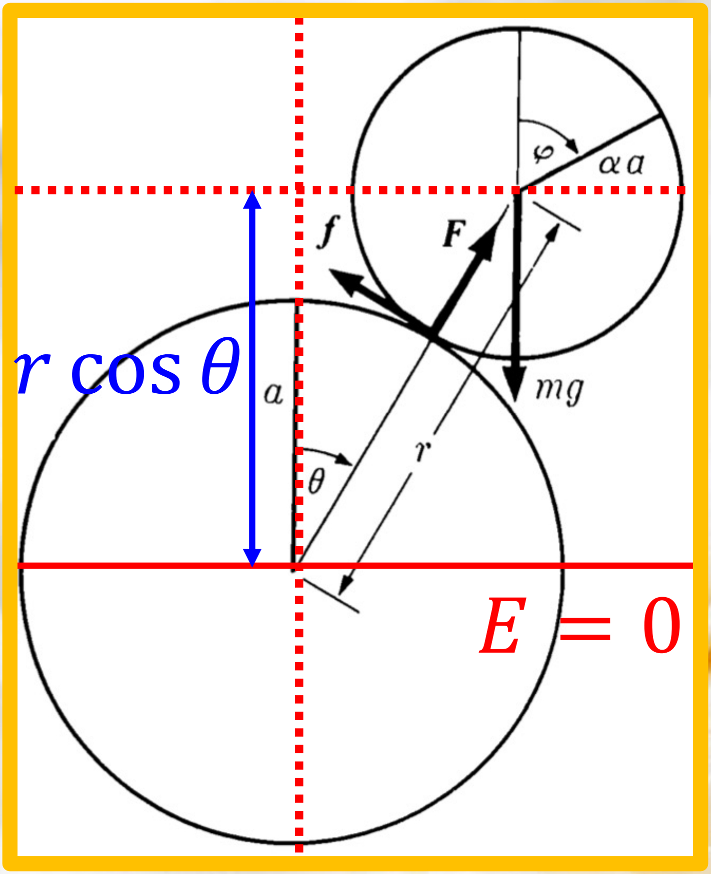
$$E = T + V = \frac{3}{4}(1+\alpha)^2 ma^2\dot{\theta}^2 + mga(1+\alpha)\cos\theta = cte.$$

که در آن انرژی کل مقدار ثابتی است

چرا؟

زیرا معادله دوم ایجاب می کند

چگونه؟



کافی است که از طرفین معادله دوم لاگرانژ نسبت به زمان انتگرال بگیریم و رابطه انرژی را به دست آوریم

$$\frac{3}{2}m(1+\alpha)^2a^2\ddot{\theta} = (1+\alpha)mga\sin\theta$$

یا به طور ساده تر از طرفین معادله انرژی نسبت به زمان مشتق بگیریم و معادله دوم لاگرانژ را به دست آوریم

$$E = \frac{3}{4}(1+\alpha)^2ma^2\dot{\theta}^2 + mga(1+\alpha)\cos\theta = cte.$$

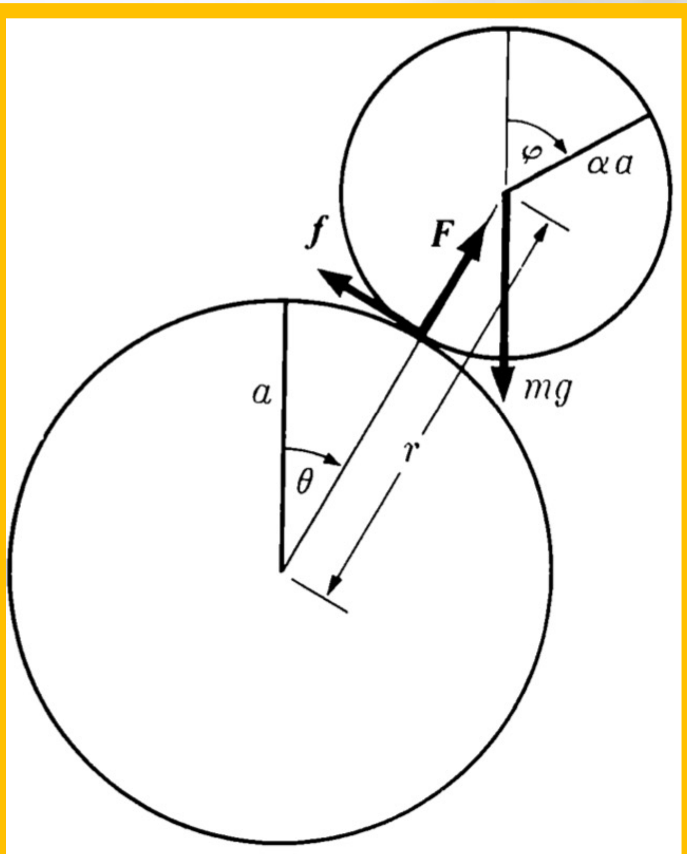
$$\frac{dE}{dt} = \frac{3}{4}(1+\alpha)^2ma^2\left(2\dot{\theta}\ddot{\theta}\right) - mga(1+\alpha)\dot{\theta}\sin\theta = 0$$

$$\frac{3}{2}(1+\alpha)^2ma^2\dot{\theta}\ddot{\theta} = mga(1+\alpha)\dot{\theta}\sin\theta$$

$$\frac{3}{2}(1+\alpha)^2ma^2\ddot{\theta} = mga(1+\alpha)\sin\theta$$

که همان معادله دوم لاگرانژ است

البته به طور فیزیکی هم انرژی باید ثابت باشد زیرا نیروی وزن پایستار است و نیروهای قیدی هم نمی توانند کار انجام دهند



$$E=\frac{3}{4}(1+\alpha)^2ma^2\dot{\theta}^2+mga(1+\alpha)\cos\theta$$

در بالاترین نقطه داریم

$$E=0+mga(1+\alpha)=mga(1+\alpha)$$

$$mga(1+\alpha)=\frac{3}{4}(1+\alpha)^2ma^2\dot{\theta}^2+mga(1+\alpha)\cos\theta$$

$$g=\frac{3}{4}(1+\alpha)a\dot{\theta}^2+g\cos\theta$$

$$\frac{3}{4}(1+\alpha)a\dot{\theta}^2=g(1-\cos\theta)$$

$$\frac{3}{4}(1+\alpha)a\dot{\theta}^2=2g\sin^2\frac{\theta}{2}$$

$$\frac{3}{4}(1+\alpha)a\dot{\theta}^2 = 2g\sin^2\frac{\theta}{2}$$

$$\dot{\theta}^2 = 4\times\frac{2}{3(1+\alpha)}\frac{g}{a}\sin^2\frac{\theta}{2}$$

$$\dot{\theta} = 2\times\sqrt{\frac{2}{3(1+\alpha)}}\sqrt{\frac{g}{a}}\sin\frac{\theta}{2}$$

$$\dot{\theta} = 2\sqrt{\beta}\sqrt{\frac{g}{a}}\sin\frac{\theta}{2}$$

$$\beta = \frac{2}{3(1+\alpha)}$$

$$\dot{\theta} = 2\sqrt{\frac{\beta g}{a}}\sin\frac{\theta}{2}$$

$$\frac{d\theta}{dt} = 2\sqrt{\frac{\beta g}{a}}\sin\frac{\theta}{2}$$

$$\frac{\frac{1}{2}d\theta}{\sin\frac{\theta}{2}} = \sqrt{\frac{\beta g}{a}}dt$$

$$\int_0^\theta\frac{\frac{1}{2}d\theta}{\sin\frac{\theta}{2}} = \sqrt{\frac{\beta g}{a}}\int_0^t dt$$

$$\int_0^\theta\frac{\frac{1}{2}d\theta}{\sin\frac{\theta}{2}} = \sqrt{\frac{\beta g}{a}}t$$

$$\int_0^{\theta} \frac{\frac{1}{2}d\theta}{\sin\frac{\theta}{2}} = \sqrt{\frac{\beta g}{a}}t$$

$$I = \int_0^{\theta} \frac{\frac{1}{2}d\theta}{\sin\frac{\theta}{2}}$$

$$\frac{\theta}{2} = x \to d\theta = 2dx$$

$$I = \int \frac{dx}{\sin x} = \int \csc x \, dx$$

$$\csc x + \cot x = u$$

$$(-\csc^2 x - \cot x \csc x)dx = du$$

$$I = \int \frac{dx}{\sin x} = -\int \frac{\csc x \, du}{\csc x \,(\csc x + \cot x)}$$

$$I = -\int \frac{du}{\csc x + \cot x} = -\int \frac{du}{u}$$

$$I = -\ln u = -\ln(\csc x + \cot x)$$

$$\csc x + \cot x = \frac{1}{\sin x} + \frac{\cos x}{\sin x}$$

$$= \frac{1 + \cos x}{\sin x} = \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} = \cot\frac{x}{2}$$

$$I = -\ln\left(\cot\frac{x}{2}\right) = \ln\left(\tan\frac{x}{2}\right)$$

$$I = \ln\left(\tan\frac{\theta}{4}\right)$$

$$\left[\ln\left(\tan\frac{\theta}{4}\right)\right]_0^{\theta} = \sqrt{\frac{\beta g}{a}}t$$

زاویه صفر به بالاترین نقطه اشاره می کند

$$\left[\ln\left(\tan\frac{\theta}{4}\right)\right]_0^{\theta} = \sqrt{\frac{\beta g}{a}}t$$

در بالاترین نقطه استوانه در تعادل است

$$\ln\left(\tan\frac{\theta}{4}\right) - \ln\left(\tan\frac{0}{4}\right) = \sqrt{\frac{\beta g}{a}}t$$

یعنی استوانه در بالاترین نقطه می تواند تا مدت طولانی (بینهایت) در حال تعادل قرار گیرد

$$\ln\left(\tan\frac{\theta}{4}\right) - \ln(0) = \sqrt{\frac{\beta g}{a}}t$$

اما این نقطه یک نقطه تعادل ناپایدار است

یعنی در عمل با یک تلنگور استوانه به پایین می غلتد

$$\ln\left(\tan\frac{\theta}{4}\right) - (-\infty!!!) = \sqrt{\frac{\beta g}{a}}t$$

پس از نظر فیزیکی مشکلی وجود ندارد

برای رفع مشکل ریاضی هم کافی است به جای زاویه صفر بسیار کوچکی اختیار کنیم

$$\ln\left(\tan\frac{\theta}{4}\right) - \ln\left(\tan\frac{\theta_0}{4}\right) = \sqrt{\frac{\beta g}{a}}t$$

$$\ln\left(\frac{\tan\frac{\theta}{4}}{\tan\frac{\theta_0}{4}}\right)=\sqrt{\frac{\beta g}{a}}t$$

$$\frac{\tan\frac{\theta}{4}}{\tan\frac{\theta_0}{4}}=\exp\left(\sqrt{\frac{\beta g}{a}}t\right)$$

$$\tan\frac{\theta}{4}=\tan\frac{\theta_0}{4}\exp\left(\sqrt{\frac{\beta g}{a}}t\right)$$

$$\tan\frac{\theta}{4}=\lim_{t\to\infty}\tan\frac{\theta_0}{4}\exp\left(\sqrt{\frac{\beta g}{a}}t\right)\to\infty\Rightarrow\frac{\theta}{4}=\frac{\pi}{2}\Rightarrow\theta=2\pi$$

یعنی اگر قیود غلتش و روی سطح باقی ماندن بتوانند پاربرجا باشند
پس از گذشت زمان بینهایت استوانه می‌تواند یک دور کامل بزند!

اما غلتش تا وقتی امکان پذیر است که معادله زیر برقرار باشد

شرط یا شروع لغزش

$$f \leq \mu F$$

با قرار دادن معادلات لاگرانژ در شرط لغزش فوق خواهیم داشت:

$$\begin{cases} F = mg\cos\theta - (1+\alpha)ma\dot{\theta}^2 \\ \dfrac{3}{2}(1+\alpha)a\ddot{\theta} = g\sin\theta \\ f = \dfrac{1}{2}(1+\alpha)ma\ddot{\theta} \end{cases}$$

$$\frac{1}{2}(1+\alpha)ma\left[\frac{2g\sin\theta_1}{3(1+\alpha)a}\right] \leq \mu\left[mg\cos\theta_1 - (1+\alpha)ma\dot{\theta}_1^2\right]$$

که در آن **تتایک** زاویه شروع لغزش

$$\theta_1$$

$$\frac{1}{3}g\sin\theta_1 \leq \mu\left[g\cos\theta_1 - (1+\alpha)a\dot{\theta}_1^2\right]$$

حال با قرار دادن رابطه

$$\dot{\theta} = 2\sqrt{\frac{\beta g}{a}\sin\frac{\theta}{2}}$$

در معادله

$$\frac{1}{3} g \sin\theta_1 \leq \mu\left[g\cos\theta_1 - (1+\alpha)a\dot{\theta}_1^2\right]$$

$$\frac{1}{3} g \sin\theta_1 \leq \mu\left[g\cos\theta_1 - (1+\alpha)a\times 4\times \frac{\beta g}{a}\sin^2\frac{\theta_1}{2}\right]$$

$$\beta = \frac{2}{3(1+\alpha)}$$

$$\frac{1}{3}\sin\theta_1 \leq \mu\left[\cos\theta_1 - (1+\alpha)\times 4\times \frac{2}{3(1+\alpha)}\frac{1-\cos\theta_1}{2}\right]$$

$$\sin\theta_1 \leq \mu[3\cos\theta_1 - 4 + 4\cos\theta_1]$$

$$\sin\theta_1 \leq \mu[7\cos\theta_1 - 4]$$

$$\boxed{\sin\theta_1 \leq \mu[7\cos\theta_1 - 4]}$$

$$\boxed{\sqrt{1-\cos^2\theta_1} \leq \mu[7\cos\theta_1 - 4]}$$

$$\boxed{1-\cos^2\theta_1 \leq \{\mu[7\cos\theta_1 - 4]\}^2}$$

$$\boxed{1-\cos^2\theta_1 \leq 49\mu^2\cos^2\theta_1 - 56\mu^2\cos\theta_1 + 16\mu^2}$$

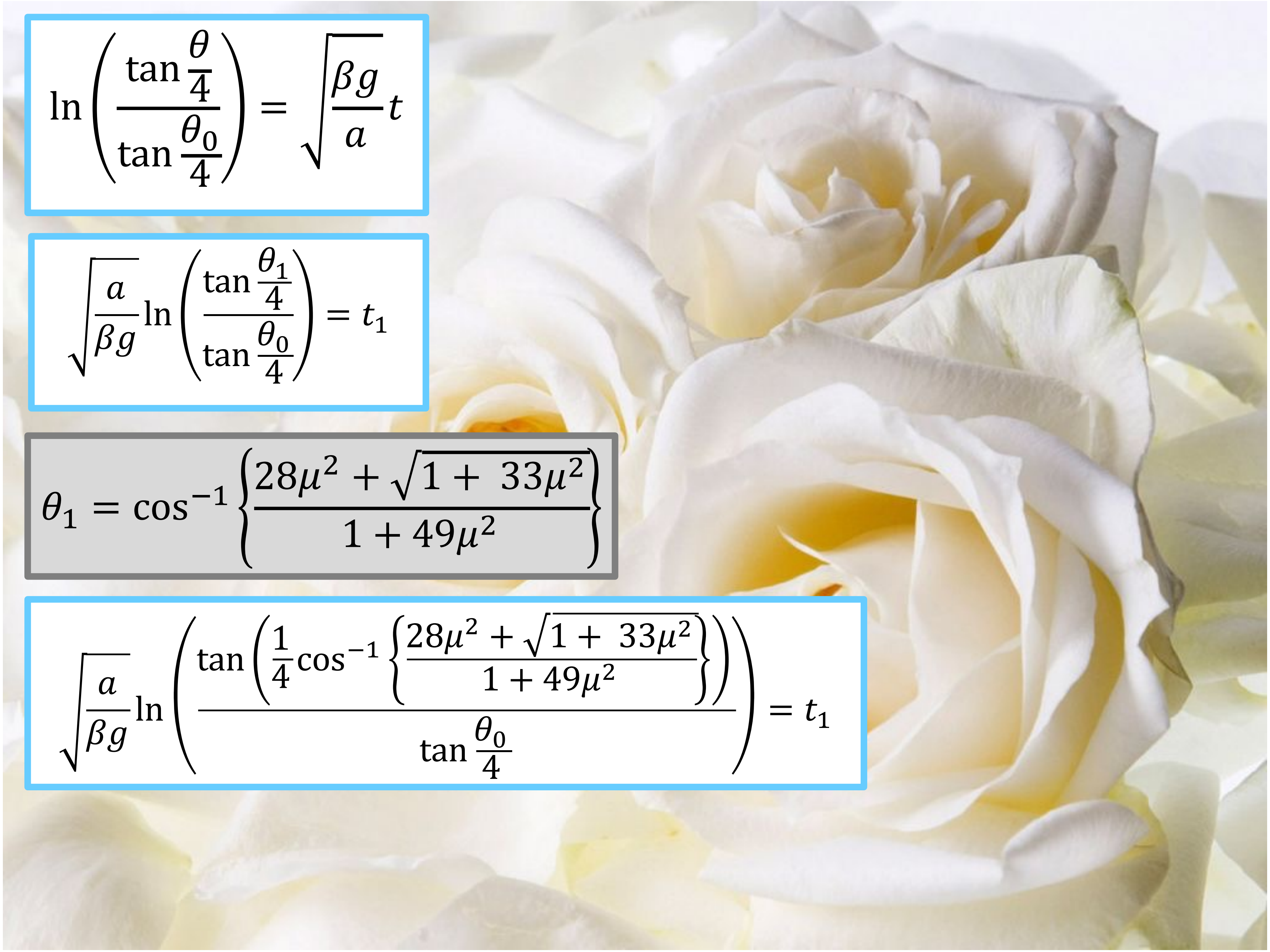
$$\boxed{0 \leq (1+49\mu^2)\cos^2\theta_1 - 56\mu^2\cos\theta_1 + 16\mu^2 - 1}$$

$$\boxed{\cos\theta_1 = \frac{28\mu^2 \pm \sqrt{28^2\mu^4 - (1+49\mu^2)(16\mu^2-1)}}{1+49\mu^2}}$$

$$\boxed{\cos\theta_1 = \frac{28\mu^2 \pm \sqrt{784\mu^4 - (16\mu^2 - 1 + 784\mu^4 - 49\mu^2)}}{1+49\mu^2}}$$

$$\boxed{\cos\theta_1 = \frac{28\mu^2 \pm \sqrt{1+33\mu^2}}{1+49\mu^2}}$$

$$\boxed{\theta_1 = \cos^{-1}\left\{\frac{28\mu^2 + \sqrt{1+33\mu^2}}{1+49\mu^2}\right\}}$$



$$\ln\left(\frac{\tan\frac{\theta}{4}}{\tan\frac{\theta_0}{4}}\right) = \sqrt{\frac{\beta g}{a}}\,t$$

$$\sqrt{\frac{a}{\beta g}}\ln\left(\frac{\tan\frac{\theta_1}{4}}{\tan\frac{\theta_0}{4}}\right) = t_1$$

$$\theta_1 = \cos^{-1}\left\{\frac{28\mu^2 + \sqrt{1 + 33\mu^2}}{1 + 49\mu^2}\right\}$$

$$\sqrt{\frac{a}{\beta g}}\ln\left(\frac{\tan\left(\frac{1}{4}\cos^{-1}\left\{\frac{28\mu^2 + \sqrt{1 + 33\mu^2}}{1 + 49\mu^2}\right\}\right)}{\tan\frac{\theta_0}{4}}\right) = t_1$$

برای بخش دوم حرکت که در آن استوانه شروع به لغزش می کند، باید از معادلات لاگرانژ اصلی زیر قبل از اعمال قیود استفاده کرد:

$$
\begin{cases}
m\ddot{r} - mr\dot{\theta}^2 = F - mg\cos\theta \\
m\left(r^2 + \frac{1}{2}a^2(1+\alpha)^2\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} - \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma} \ = mgr\sin\theta \\
-\frac{1}{2}ma^2(1+\alpha)^2\ddot{\theta} + \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma} \ = -fr
\end{cases}
$$

چون اصطحکاک سبب غلتش می شود و ما لغزش می خواهیم، در معادلات سوم لاگرانژ نیروی اصطحکاک را بربر صفر قرار می دهیم

$$\boxed{f = 0}$$

چون استوانه باید تماس خود را با سطح استوانه زیرین حفظ کند پس قید زیر را هم اعمال می کنیم:

$$\boxed{r = a(1+\alpha) = \text{cte.}} \qquad \boxed{\dot{r} = 0} \qquad \boxed{\ddot{r} = 0}$$

$$\begin{cases} -ma(1+\alpha)\dot{\theta}^2 = F - mg\cos\theta \\ m\left(\frac{3}{2}a^2(1+\alpha)^2\right)\ddot{\theta} - \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma} = mga(1+\alpha)\sin\theta \\ -\frac{1}{2}ma^2(1+\alpha)^2\ddot{\theta} + \frac{1}{2}ma^2(1+\alpha)^2\ddot{\gamma} = 0 \end{cases}$$

$$\begin{cases} F = mg\cos\theta - ma(1+\alpha)\dot{\theta}^2 \\ \frac{3}{2}a(1+\alpha)\ddot{\theta} - \frac{1}{2}a(1+\alpha)\ddot{\gamma} = g\sin\theta \\ -\ddot{\theta} + \ddot{\gamma} = 0 \end{cases}$$

$$-\ddot{\theta} + \ddot{\gamma} = 0$$

$$\ddot{\theta} = \ddot{\gamma}$$

$$\frac{3}{2}a(1+\alpha)\ddot{\theta} - \frac{1}{2}a(1+\alpha)\ddot{\theta} = g\sin\theta$$

$$a(1+\alpha)\ddot{\theta} = g\sin\theta$$

$$F = 0$$

معادلات لاگرانژ در لحظه جدایی

$$\begin{cases} g\cos\theta = a(1+\alpha)\dot{\theta}^2 \\ a(1+\alpha)\ddot{\theta} = g\sin\theta \end{cases}$$

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \dot{\theta}\frac{d\dot{\theta}}{dt}\frac{dt}{d\theta} = \dot{\theta}\ddot{\theta}\frac{1}{\dot{\theta}} = \ddot{\theta}$$

$$\begin{cases} g\cos\theta = a(1+\alpha)\dot{\theta}^2 \\ a(1+\alpha)\dot{\theta}\frac{d\dot{\theta}}{d\theta} = g\sin\theta \end{cases}$$

$$\begin{cases} g\cos\theta = a(1+\alpha)\dot{\theta}^2 \\ a(1+\alpha)\dot{\theta}\dfrac{d\dot{\theta}}{d\theta} = g\sin\theta \end{cases}$$

$$a(1+\alpha)\dot{\theta}d\dot{\theta} = g\sin\theta\, d\theta$$

$$a(1+\alpha)\int_{\theta_1}^{\theta_2}\dot{\theta}d\dot{\theta} = g\int_{\theta_1}^{\theta_2}\sin\theta\, d\theta$$

$$a(1+\alpha)\frac{1}{2}\left(\dot{\theta}_2^2 - \dot{\theta}_1^2\right) = -g(\cos\theta_2 - \cos\theta_1)$$

$$\begin{cases} g\cos\theta_2 = a(1+\alpha)\dot{\theta}_2^2 \\ a(1+\alpha)\dfrac{1}{2}\left(\dot{\theta}_2^2 - \dot{\theta}_1^2\right) = -g(\cos\theta_2 - \cos\theta_1) \end{cases}$$

$$\begin{cases} a(1+\alpha)\dot{\theta}_2^2 = g\cos\theta_2 \\ a(1+\alpha)\left(\dot{\theta}_2^2 - \dot{\theta}_1^2\right) = -2g\cos\theta_2 + 2g\cos\theta_1 \end{cases}$$

$$a(1+\alpha)\dot{\theta}_1^2 = 3g\cos\theta_2 - 2g\cos\theta_1$$

که در آن تتادو زاویه جدایی از سطح که به صورت زیر به دست می‌آید

$$a(1+\alpha)\dot{\theta}_1^2 = 3g\cos\theta_2 - 2g\cos\theta_1$$

$$\cos\theta_2 = \frac{a(1+\alpha)}{3g}\dot{\theta}_1^2 + \frac{2}{3}\cos\theta_1$$

$$\theta_2 = \cos^{-1}\left[\frac{a(1+\alpha)}{3g}\dot{\theta}_1^2 + \frac{2}{3}\cos\theta_1\right]$$

$$\dot{\theta}_1 = 2\sqrt{\frac{\beta g}{a}}\sin\frac{\theta_1}{2}$$

$$\theta_1 = \cos^{-1}\left\{\frac{28\mu + \sqrt{1+\ 33\mu^2}}{1+49\mu^2}\right\}$$

$$\beta = \frac{2}{3(1+\alpha)}$$

$$\theta_2 = \cos^{-1}\left[\frac{a(1+\alpha)}{3g}\left(4\frac{2g}{3a(1+\alpha)}\right)\sin\frac{1}{2}\left[\cos^{-1}\left\{\frac{28\mu + \sqrt{1+\ 33\mu^2}}{1+49\mu^2}\right\}\right] + \frac{2}{3}\frac{28\mu + \sqrt{1+\ 33\mu^2}}{1+49\mu^2}\right]$$

$$\theta_2 = \cos^{-1}\left[\frac{8}{3}\sin\frac{1}{2}\left[\cos^{-1}\left\{\frac{28\mu + \sqrt{1+\ 33\mu^2}}{1+49\mu^2}\right\}\right] + \frac{2}{3}\frac{28\mu + \sqrt{1+\ 33\mu^2}}{1+49\mu^2}\right]$$

تست درستی نتیجه

$$\mu = 0 \ \rightarrow \theta_2 = \cos^{-1}\left(\frac{8}{3}\times 0 + \frac{2}{3}\times 1\right) = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\int_{\dot\theta_1}^{\dot\theta} \frac{d\dot\theta}{\dot\theta\sqrt{4\beta g a - a^2\dot\theta^2}} = \frac{3}{4a}t$$

$$\int_{\dot\theta_1}^{\dot\theta} \frac{d\dot\theta}{\dot\theta a\sqrt{\frac{4\beta g}{a} - \dot\theta^2}} = \frac{3}{4a}t$$

$$\dot\theta = \sqrt{\frac{4\beta g}{a}}\operatorname{sech} u$$

$$d\dot\theta = -\sqrt{\frac{4\beta g}{a}}\operatorname{sech} u \tanh u\, du$$

$$\int \frac{-\sqrt{\frac{4\beta g}{a}}\operatorname{sech} u \tanh u\, du}{a\sqrt{\frac{4\beta g}{a}}\operatorname{sech} u\sqrt{\frac{4\beta g}{a} - \frac{4\beta g}{a}\sin^2 u}} = \frac{3}{4a}t$$

$$\int \frac{-\tanh u\, du}{a\sqrt{\frac{4\beta g}{a}}\sqrt{1 - \operatorname{sech}^2 u}} = \frac{3}{4a}t$$

$$\int \frac{-\tanh u\, du}{\sqrt{4a\beta g}\tanh u} = \frac{3}{4a}t$$

$$\int \frac{du}{2\sqrt{a\beta g}} = \frac{3}{4a}t$$

$$\int du = \frac{3}{4a}2\sqrt{a\beta g}\,t$$

$$u = \frac{3}{2}\sqrt{\frac{\beta g}{a}}t$$

$$\operatorname{sech}^{-1}\left(\sqrt{\frac{a}{4\beta g}}\dot\theta\right) = \frac{3}{2}\sqrt{\frac{\beta g}{a}}t$$

$$\operatorname{sech}^{-1}\left(\sqrt{\frac{a}{4\beta g}}\dot{\theta}\right)=\frac{3}{2}\sqrt{\frac{\beta g}{a}}t$$

$$\sqrt{\frac{a}{4\beta g}}\dot{\theta}=\operatorname{sech}\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t\right)$$

$$\dot{\theta}=2\sqrt{\frac{\beta g}{a}}\operatorname{sech}\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t\right)$$

$$d\theta=2\sqrt{\frac{\beta g}{a}}\operatorname{sech}\left(3\sqrt{\frac{\beta g}{a}}t\right)\mathrm{dt}$$

$$\theta_2-\theta_1=2\sqrt{\frac{\beta g}{a}}\int_{t_1}^{t_2}\operatorname{sech}\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t\right)\mathrm{dt}$$

$$\frac{3}{2}\sqrt{\frac{\beta g}{a}}t=z$$

$$\mathrm{dt}=\frac{2}{3}\sqrt{\frac{a}{\beta g}}dz$$

$$\theta_2-\theta_1=2\sqrt{\frac{\beta g}{a}}\frac{2}{3}\sqrt{\frac{a}{\beta g}}\int_{t_1}^{t_2}\operatorname{sech}\mathrm{z}\,\mathrm{dz}$$

$$\theta_2-\theta_1=\frac{4}{3}\int\operatorname{sech}\mathrm{z}\,\mathrm{dz}$$

$$\operatorname{sech}\mathrm{z}=\frac{1}{\cosh z}=\frac{\cosh z}{\cosh^2 z}=\frac{\cosh z}{1+\sinh^2 z}$$

$$\theta_2-\theta_1=\frac{4}{3}\int\frac{\cosh z\,\mathrm{dz}}{1+\sinh^2 z}$$

$$\sinh z=w$$

$$\cosh z\,dz=dw$$

$$\theta_2-\theta_1=\frac{4}{3}\int\frac{\mathrm{dw}}{1+w^2}=\frac{4}{3}\tan^{-1}w$$

$$\theta_2-\theta_1=\frac{4}{3}\tan^{-1}\sinh z$$

$$=\frac{4}{3}\tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t\right)$$

که در آن t_2، زمان کل از شروع غلتش تا جدایی از سطح استوانه زیرن، به صورت زیر به دست می‌آید

$$\theta_2-\theta_1=\left[\frac{4}{3}\tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t\right)\right]_{t_1}^{t_2}$$

$$\theta_2-\theta_1=\frac{4}{3}\tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t_2\right)-\frac{4}{3}\tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t_1\right)$$

$$\theta_2-\theta_1+\frac{4}{3}\tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t_1\right)=\frac{4}{3}\tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t_2\right)$$

$$\tan\left[\frac{3}{4}(\theta_2-\theta_1)+\tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t_1\right)\right]=\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t_2\right)$$

$$\frac{2}{3}\sqrt{\frac{a}{\beta g}}\sinh^{-1}\left[\tan\left[\frac{3}{4}(\theta_2-\theta_1)+\tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t_1\right)\right]\right]=t_2$$

$$t_2 = \frac{2}{3}\sqrt{\frac{a}{\beta g}}\sinh^{-1}\left[\tan\left[\frac{3}{4}(\theta_2 - \theta_1) + \tan^{-1}\sinh\left(\frac{3}{2}\sqrt{\frac{\beta g}{a}}t_1\right)\right]\right]$$

$$\theta_1 = \cos^{-1}\left\{\frac{28\mu + \sqrt{1 + 33\mu^2}}{1 + 49\mu^2}\right\}$$

$$\sqrt{\frac{a}{\beta g}}\ln\left(\frac{\tan\frac{\theta_1}{4}}{\tan\frac{\theta_0}{4}}\right) = t_1$$

$$\beta = \frac{2}{3(1+\alpha)}$$

$$\theta_2 = \cos^{-1}\left[\frac{8}{3}\sin\frac{1}{2}\left[\cos^{-1}\left\{\frac{28\mu + \sqrt{1 + 33\mu^2}}{1 + 49\mu^2}\right\}\right] + \frac{2}{3}\frac{28\mu + \sqrt{1 + 33\mu^2}}{1 + 49\mu^2}\right]$$