بسم الله الرحمن الرحیم

تمرین‌های مکانیک تحلیلی ۲

تهیه شده توسط دکتر لیلا ملاباشی
پژوهشگر پسادکتری

۱. مختصات u و w طبق معادلات زیر برحسب مختصات قطبی r و θ تعریف شده‌اند

$$u = \ln(r/a) - \theta\cot\zeta\,,$$

$$w = \ln(r/a) + \theta\tan\zeta\,,$$

که در آن a و ζ ثابت‌اند. منحنیهای u ثابت و منحنیهای w ثابت را رسم کنید. انرژی جنبشی ذره‌ای به جرم m را برحسب u، w و $\dot{u}$، $\dot{w}$ پیدا کنید. عباراتی برای Q_u و Q_w برحسب مؤلفه‌های نیروی قطبی F_r و F_θ به‌دست آورید. P_u و P_w را پیدا کنید. مطلوب است نیروهای Q_u و Q_w که برای حرکت دادن ذره با تندی ثابت $\dot{s}$ در امتداد مارپیچی با u ثابت ($u = u_0$)، لازم است.

اگر u ثابت باشد.

$u = c$

$$c = \ln\left(\frac{r}{a}\right) - \theta\cot\zeta$$

$$\ln\left(\frac{r}{a}\right) = c + \theta\cot\zeta$$

$$\frac{r}{a} = e^{(c+\theta\cot\zeta)}$$

$$r = \mathrm{a}\, e^{(c+\theta\cot\zeta)}$$

$$r = \mathrm{a}\, e^{c}\, e^{\theta\cot\zeta}$$

$$r = A\, e^{\theta\cot\zeta}$$

```
In [1]: import sympy as smp
        from sympy import *

In [2]: smp.init_printing()

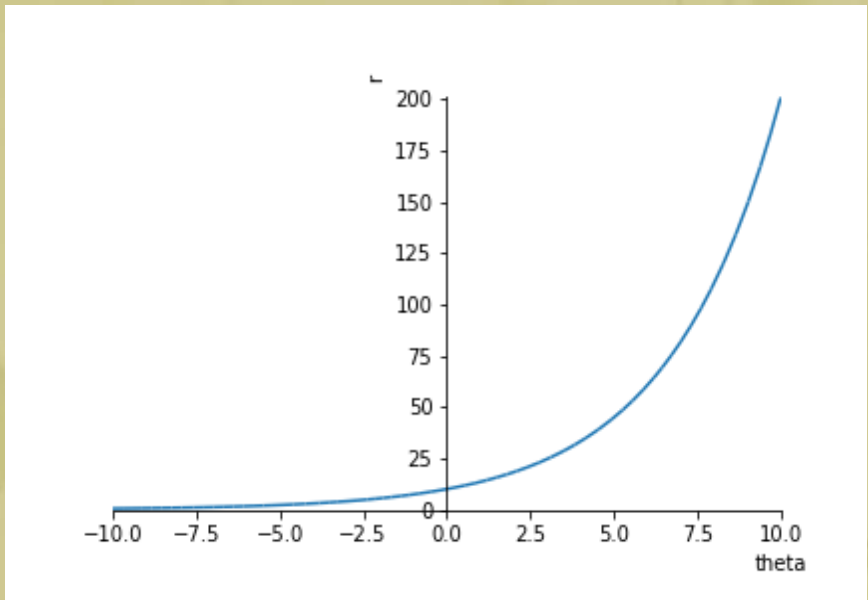
In [3]: A = smp.symbols('A')
        r = smp.symbols("r")
        theta = smp.symbols('theta')
        zeta = smp.symbols('zeta')

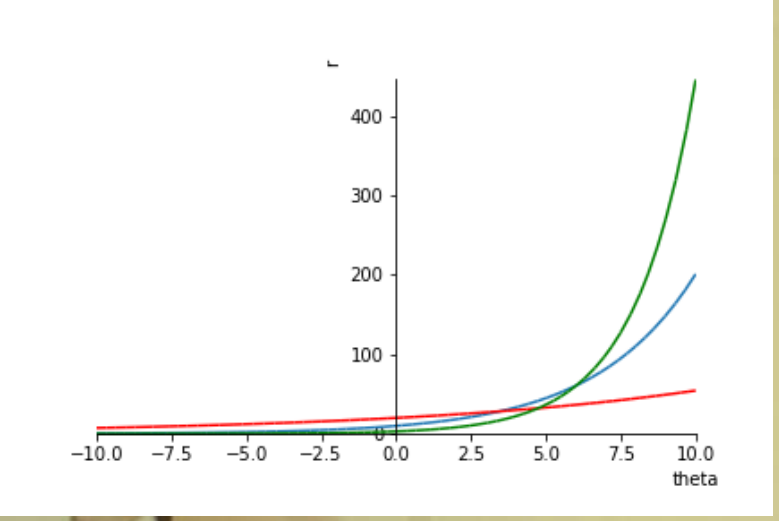
In [4]: r = A*smp.exp(theta*smp.cot(zeta))

In [5]: r
Out[5]: A e^{θ cot(ζ)}

In [7]: graph = smp.plot(10*smp.exp(0.3*theta), ylabel="r", show=True)
        graph.save('mechanic1_1')
```

با فرض $A = 10$ و $\cot\zeta = 0.3$



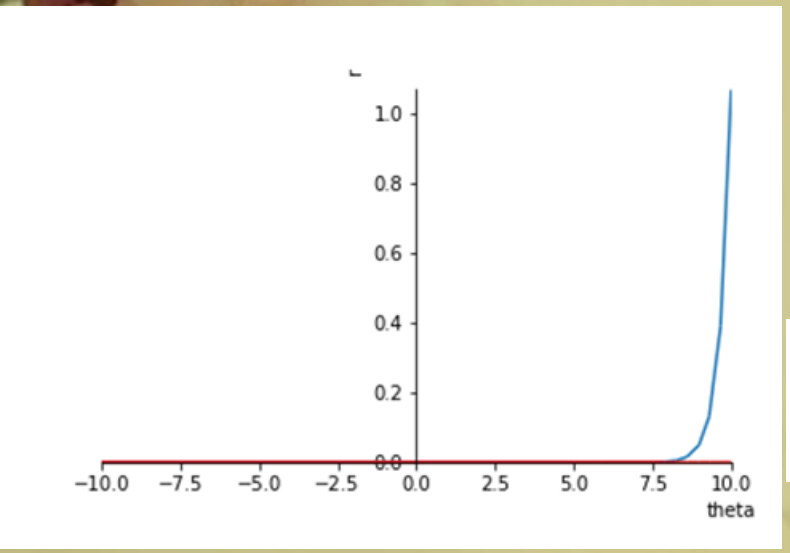


با فرض

$A = 20$ و $\cot\zeta = 0.1$

$A = 3$ و $\cot\zeta = 0.5$

```
In [18]: graph = smp.plot(10*smp.exp(0.3*theta), ylabel="r", show=False)
         graph1 = smp.plot(20*smp.exp(0.1*theta), ylabel="r", line_color = 'red', show=False)
         graph2 = smp.plot(3*smp.exp(0.5*theta), ylabel="r", line_color = 'green', show=False)
         graph.append(graph1[0])
         graph.append(graph2[0])
         graph.save('mechanic1_1')
         graph.show()
```



با فرض

$A = 10$ و $\cot\zeta = 3$

$A = 10$ و $\cot\zeta = 2$

```
graph = smp.plot(10*smp.exp(3*theta), ylabel="r", show=False)
graph1 = smp.plot(10*smp.exp(2*theta), ylabel="r", line_color = 'red', show=False)
graph.append(graph1[0])
graph.save('mechanic1_1')
graph.show()
```

اگر w ثابت باشد.

$w = d$

$$d = \ln\left(\frac{r}{a}\right) + \theta\tan\zeta$$

$$\ln\left(\frac{r}{a}\right) = d - \theta\tan\zeta$$

$$\frac{r}{a} = e^{(d-\theta\tan\zeta)}$$

$$r = \mathrm{a}\, e^{(d-\theta\mathrm{tant}\,\zeta)}$$

$$r = \mathrm{a}\, e^{d}\, e^{-\theta\tan\zeta}$$

$$r = B\, e^{-\theta\tan\zeta}$$

با فرض $B = 15$ و $\tan\zeta = 10/3$

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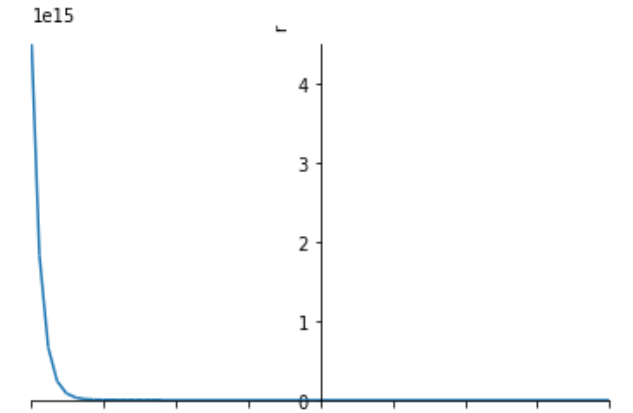
In [8]: graph = smp.plot(15*smp.exp(-10/3*theta), ylabel="r", show=True)
        graph.save('mechanic1_2')
```



انرژی جنبشی ذره‌ای به جرم m را برحسب u، w و $\dot{u}$، $\dot{w}$ پیدا کنید.

$$T = \frac{1}{2}\left(m\dot{r}^2 + mr^2\dot{\theta}^2\right)$$

$$\tan\zeta \times u = \tan\zeta \times \ln\left(\frac{r}{a}\right) - \theta$$

$$\cot\zeta \times w = \cot\zeta \times \ln\left(\frac{r}{a}\right) + \theta$$

$$u = \ln\left(\frac{r}{a}\right) - \theta\cot\zeta$$

$$w = \ln\left(\frac{r}{a}\right) + \theta\tan\zeta$$

$$u\tan\zeta + w\cot\zeta = \tan\zeta\ln\left(\frac{r}{a}\right) + \cot\zeta\ln\left(\frac{r}{a}\right)$$

$$u\tan\zeta + w\cot\zeta = (\tan\zeta + \cot\zeta)\ln\left(\frac{r}{a}\right)$$

$$\ln\left(\frac{r}{a}\right) = \frac{u\,tan\zeta + w\cot\zeta}{(tan\zeta + cot\zeta)}$$

$$u\,tan\zeta + w\cot\zeta = u\frac{sin\zeta}{cos\zeta} + w\frac{cos\zeta}{sin\zeta} = \frac{usin^2\zeta + wcos^2\zeta}{cos\zeta\,sin\zeta}$$

$$\frac{1}{(\tan\zeta + \cot\zeta)} = \frac{1}{\frac{\sin\zeta}{\cos\zeta} + \frac{\cos\zeta}{\sin\zeta}} = \frac{1}{\frac{\sin^2\zeta + \cos^2\zeta}{\sin\zeta\cos\zeta}} = \sin\zeta\cos\zeta$$

$$\ln\left(\frac{r}{a}\right) = \frac{u\,tan\zeta + w\cot\zeta}{(tan\zeta + cot\zeta)} = \frac{usin^2\zeta + wcos^2\zeta}{\cos\zeta\ \sin\zeta} \times \sin\zeta\cos\zeta$$

$$\ln\left(\frac{r}{a}\right) = usin^2\zeta + wcos^2\zeta$$

$$\frac{r}{a} = e^{usin^2\zeta + wcos^2\zeta}$$

$$r = ae^{u\sin^2\zeta + w\cos^2\zeta}$$

$$r = ae^{u\sin^2\zeta + w\cos^2\zeta}$$

$$\dot{r} = a(\dot{u}\sin^2\zeta + \dot{w}\cos^2\zeta)e^{u\sin^2\zeta + w\cos^2\zeta}$$

$$u = \ln\left(\frac{r}{a}\right) - \theta\cot\zeta$$

$$w = \ln\left(\frac{r}{a}\right) + \theta\tan\zeta$$

$$w - u = \theta(\tan\zeta + \cot\zeta)$$

$$\theta = \frac{w-u}{\tan\zeta + \cot\zeta}$$

$$\frac{1}{(\tan\zeta + \cot\zeta)} = \sin\zeta\cos\zeta$$

$$\theta = (w-u)\sin\zeta\cos\zeta$$

$$\dot{\theta} = (\dot{w} - \dot{u})\sin\zeta\cos\zeta$$

$$T = \frac{1}{2}\left(m\dot{r}^2 + mr^2\dot{\theta}^2\right)$$

$$r = ae^{u\sin^2\zeta + w\cos^2\zeta}$$

$$\dot{r} = a(\dot{u}\sin^2\zeta + \dot{w}\cos^2\zeta)e^{u\sin^2\zeta + w\cos^2\zeta}$$

$$\dot{\theta} = (\dot{w} - \dot{u})\sin\zeta\cos\zeta$$

$$T = \frac{1}{2}m\left(a^2(\dot{u}\sin^2\zeta + \dot{w}\cos^2\zeta)^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)} + a^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)}(\dot{w} - \dot{u})^2\sin^2\zeta\cos^2\zeta\right)$$

$$T = \frac{1}{2}ma^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)}[(\dot{u}\sin^2\zeta + \dot{w}\cos^2\zeta)^2 + (\dot{w} - \dot{u})^2\sin^2\zeta\cos^2\zeta]$$

$$(\dot{u}\sin^2\zeta + \dot{w}\cos^2\zeta)^2 = (\dot{u}\sin^2\zeta)^2 + (\dot{w}\cos^2\zeta)^2 + 2\dot{u}\sin^2\zeta\,\dot{w}\cos^2\zeta$$

$$(\dot{w} - \dot{u})^2\sin^2\zeta\cos^2\zeta = \dot{w}^2\sin^2\zeta\cos^2\zeta + \dot{u}^2\sin^2\zeta\cos^2\zeta - 2\dot{u}\sin^2\zeta\,\dot{w}\cos^2\zeta$$

با جمع دو رابطه بالا خواهیم داشت:

$$(\dot{u}\sin^2\zeta)^2 + (\dot{w}\cos^2\zeta)^2 + \dot{w}^2\sin^2\zeta\cos^2\zeta + \dot{u}^2\sin^2\zeta\cos^2\zeta$$

$$\dot{u}^2\sin^2\zeta(\sin^2\zeta + \cos^2\zeta) + \dot{w}^2\cos^2\zeta(\cos^2\zeta + \sin^2\zeta)$$

$$\dot{u}^2\sin^2\zeta + \dot{w}^2\cos^2\zeta$$

$$T = \frac{1}{2}\left(m\dot{r}^2 + mr^2\dot{\theta}^2\right)$$

$$T = \frac{1}{2}ma^2e^{2(usin^2\,\zeta + w\cos^2\zeta)}[(\dot{u}\sin^2\zeta + \dot{w}\cos^2\zeta)^2 + (\dot{w} - \dot{u})^2\sin^2\zeta\cos^2\zeta]$$

$$T = \frac{1}{2}ma^2e^{2(u\sin^2\zeta + w\cos^2\zeta)}(\dot{u}^2\sin^2\zeta + \dot{w}^2\cos^2\zeta)$$

عباراتی برای Q_u و Q_w برحسب مؤلفه‌های نیروی قطبی F_r و F_θ به‌دست آورید. p_u و p_w را پیدا کنید.

$$Q_u = Q_r \frac{\partial r}{\partial u} + Q_\theta \frac{\partial \theta}{\partial u}$$

$$Q_u = F_r \frac{\partial r}{\partial u} + r F_\theta \frac{\partial \theta}{\partial u}$$

$$r = a e^{u \sin^2 \zeta + w \cos^2 \zeta}$$

$$\theta = (w - u) \sin \zeta \cos \zeta$$

$$\frac{\partial r}{\partial u} = a \sin^2 \zeta \, e^{u \sin^2 \zeta + w \cos^2 \zeta}$$

$$\frac{\partial \theta}{\partial u} = -\sin \zeta \cos \zeta$$

$$Q_u = F_r a \sin^2 \zeta \, e^{u \sin^2 \zeta + w \cos^2 \zeta} - a e^{u \sin^2 \zeta + w \cos^2 \zeta} F_\theta \sin \zeta \cos \zeta$$

$$Q_u = a \sin \zeta \; e^{u \sin^2 \zeta + w \cos^2 \zeta} (F_r \sin \zeta - F_\theta \cos \zeta)$$

$$Q_w = Q_r \frac{\partial r}{\partial w} + Q_\theta \frac{\partial \theta}{\partial w}$$

$$Q_w = F_r \frac{\partial r}{\partial w} + r F_\theta \frac{\partial \theta}{\partial w}$$

$$r = a e^{u \sin^2 \zeta + w \cos^2 \zeta}$$

$$\theta = (w - u) \sin \zeta \cos \zeta$$

$$\frac{\partial r}{\partial w} = a \cos^2 \zeta e^{u \sin^2 \zeta + w \cos^2 \zeta}$$

$$\frac{\partial \theta}{\partial w} = \sin \zeta \cos \zeta$$

$$Q_w = F_r a \cos^2 \zeta e^{u \sin^2 \zeta + w \cos^2 \zeta} + a e^{u \sin^2 \zeta + w \cos^2 \zeta} F_\theta \sin \zeta \cos \zeta$$

$$Q_w = a \cos \zeta \; e^{u \sin^2 \zeta + w \cos^2 \zeta} (F_r \sin \zeta + F_\theta \cos \zeta)$$

$$p_u = \frac{\partial T}{\partial \dot{u}}$$

$$p_w = \frac{\partial T}{\partial \dot{w}}$$

$$T = \frac{1}{2} m a^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)} (\dot{u}^2 \sin^2\zeta + \dot{w}^2 \cos^2\zeta)$$

$$p_u = \frac{1}{2} m a^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)} (2\dot{u}\sin^2\zeta)$$

$$p_u = m a^2 \dot{u} \sin^2\zeta e^{2(u\sin^2\zeta + w\cos^2\zeta)}$$

$$p_w = \frac{1}{2} m a^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)} (2\dot{w}\cos^2\zeta)$$

$$p_w = m a^2 \dot{w} \cos^2\zeta e^{2(u\sin^2\zeta + w\cos^2\zeta)}$$

مطلوب است نیروهای Q_u و Q_w که برای حرکت دادن ذره با تندی ثابت $\dot{s}$ در امتداد مارپیچی با u ثابت ($u = u_0$)، لازم است.

$$T = \frac{1}{2} m a^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)} (\dot{u}^2 \sin^2\zeta + \dot{w}^2 \cos^2\zeta)$$

چون $u = u_0$ ثابت است پس $\dot{u} = \ddot{u} = 0$

$$T = \frac{1}{2} m a^2 \dot{w}^2 \cos^2\zeta e^{2(u\sin^2\zeta + w\cos^2\zeta)}$$

$$T = \frac{1}{2} m \dot{s}^2$$

$$\dot{s}^2 = a^2 \dot{w}^2 \cos^2\zeta \, e^{2(u\sin^2\zeta + w\cos^2\zeta)}$$

$$\dot{s} = a\dot{w} \cos\zeta \, e^{(u\sin^2\zeta + w\cos^2\zeta)}$$

$$\dot{s} = a\dot{w}\cos\zeta\, e^{(u\sin^2\zeta + w\cos^2\zeta)}$$

تندی ذره ثابت است پس $\ddot{s} = 0$

$$\ddot{s} = a\ddot{w}\cos\zeta\, e^{(u\sin^2\zeta + w\cos^2\zeta)} + a\dot{w}\cos\zeta\,(\dot{w}\cos^2\zeta)e^{(u\sin^2\zeta + w\cos^2\zeta)} = 0$$

$$a\ddot{w}\cos\zeta\, e^{(u\sin^2\zeta + w\cos^2\zeta)} = -\,a\dot{w}\cos\zeta\,(\dot{w}\cos^2\zeta)e^{(u\sin^2\zeta + w\cos^2\zeta)}$$

$$\ddot{w} = -\dot{w}^2\cos^2\zeta$$

$$Q_u = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{u}}\right] - \frac{\partial T}{\partial u}$$

$$T = \frac{1}{2}ma^2e^{2(u\sin^2\zeta + w\cos^2\zeta)}(\dot{u}^2\sin^2\zeta + \dot{w}^2\cos^2\zeta)$$

$$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}ma^2e^{2(u\sin^2\zeta + w\cos^2\zeta)} \times (2\dot{u}\sin^2\zeta) = 0$$

$$T = \frac{1}{2} m a^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)} (\dot{u}^2 \sin^2\zeta + \dot{w}^2 \cos^2\zeta)$$

$$\frac{\partial T}{\partial u} = \frac{1}{2} m a^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)} \times 2\sin^2\zeta \times (\dot{u}^2 \sin^2\zeta + \dot{w}^2 \cos^2\zeta)$$

$$Q_u = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{u}}\right]^0 - \frac{\partial T}{\partial u} = -\frac{1}{2} m a^2 e^{2(u\sin^2\zeta + w\cos^2\zeta)} 2\sin^2\zeta (\dot{w}^2 \cos^2\zeta)$$

$$Q_u = -m a^2 \dot{w}^2 \cos^2\zeta \sin^2\zeta e^{2(u\sin^2\zeta + w\cos^2\zeta)}$$

$$\dot{s} = a\dot{w}\cos\zeta\, e^{(u\sin^2\zeta + w\cos^2\zeta)}$$

$$Q_u = -m\dot{s}^2 \sin^2\zeta$$

$$T = \frac{1}{2} m a^2 \dot{w}^2 \cos^2 \zeta e^{2(u \sin^2 \zeta + w \cos^2 \zeta)}$$

$$Q_w = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{w}}\right] - \frac{\partial T}{\partial w}$$

$$\frac{\partial T}{\partial \dot{w}} = m a^2 \dot{w} \cos^2 \zeta e^{2(u \sin^2 \zeta + w \cos^2 \zeta)}$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{w}}\right] = m a^2 \ddot{w} \cos^2 \zeta e^{2(u \sin^2 \zeta + w \cos^2 \zeta)} + 2 \cos^2 \zeta \dot{w} m a^2 \dot{w} \cos^2 \zeta e^{2(u \sin^2 \zeta + w \cos^2 \zeta)}$$

$$\frac{\partial T}{\partial w} = \frac{1}{2} m a^2 \dot{w}^2 \cos^2 \zeta \times 2 \cos^2 \zeta e^{2(u \sin^2 \zeta + w \cos^2 \zeta)}$$

$$Q_w = m a^2 \ddot{w} \cos^2 \zeta e^{2(u \sin^2 \zeta + w \cos^2 \zeta)} + \cos^2 \zeta \dot{w} m a^2 \dot{w} \cos^2 \zeta e^{2(u \sin^2 \zeta + w \cos^2 \zeta)}$$

$$Q_w = ma^2\ddot{w}\cos^2\zeta e^{2(u\sin^2\zeta + w\cos^2\zeta)} + \cos^2\zeta\dot{w}ma^2\dot{w}\cos^2\zeta e^{2(u\sin^2\zeta + w\cos^2\zeta)}$$

$$Q_w = ma^2\cos^2\zeta e^{2(u\sin^2\zeta + w\cos^2\zeta)}[\ddot{w} + \dot{w}^2\cos^2\zeta]$$

$$\ddot{w} = -\dot{w}^2\cos^2\zeta$$

$$Q_w = ma^2\cos^2\zeta e^{2(u\sin^2\zeta + w\cos^2\zeta)}[-\dot{w}^2\cos^2\zeta + \dot{w}^2\cos^2\zeta]$$

$$Q_w = 0$$

ذره‌ای به جرم m در میدان نیروی مرکزی $F(r)=\frac{-k}{r^2}$، $(k>0)$ حرکت می‌کند. اگر (r,θ,φ) و (P_r,P_θ,P_φ) به‌ترتیب مختصات و تکانه‌های تعمیم‌یافته متناظر در دستگاه مختصات کروی باشند، کدام رابطه $\left(\dot{A}\equiv\frac{dA}{dt}\right)$ درست است؟

۱) $\dot{P}_\theta=\frac{P_\varphi^2\cot\theta}{2mr^3\sin^2\theta}$

۲) $\dot{P}_r=\frac{-k}{r^2}+\frac{P_\theta^2}{2mr^3}+\frac{P_\varphi^2}{2mr^3\sin^2\theta}$

۳) $\dot{P}_\theta=\frac{P_\varphi^2\cot\theta}{mr^3\sin^2\theta}$

۴) $\dot{P}_r=\frac{-k}{r^2}+\frac{P_\theta^2}{mr^3}+\frac{P_\varphi^2}{mr^3\sin^2\theta}$

$$T=\frac{1}{2}\left(m\dot{r}^2+mr^2\dot{\theta}^2+r^2\sin^2\theta\dot{\varphi}^2\right)$$

$$p_r=\frac{\partial T}{\partial\dot{r}}=m\dot{r}$$

$$p_\theta=\frac{\partial T}{\partial\dot{\theta}}=mr^2\dot{\theta}$$

$$p_\varphi=\frac{\partial T}{\partial\dot{\varphi}}=mr^2\sin^2\theta\dot{\varphi}$$

$$T = \frac{1}{2}\left(m\dot{r}^2 + mr^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right)$$

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$p_\varphi = \frac{\partial T}{\partial \dot{\varphi}} = mr^2\sin^2\theta\dot{\varphi}$$

$$Q_r = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{r}}\right] - \frac{\partial T}{\partial r}$$

$$F_r = \frac{d}{dt}[p_r] - \frac{\partial T}{\partial r} = -\frac{k}{r^2}$$

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2 + r\sin^2\theta\,\dot{\varphi}^2$$

$$-\frac{k}{r^2} = \dot{p}_r - \left(mr\dot{\theta}^2 + r\sin^2\theta\,\dot{\varphi}^2\right)$$

$$\dot{p}_r = -\frac{k}{r^2} + \frac{{p_\theta}^2}{mr^3} + \frac{{p_\varphi}^2}{mr^3\sin^2\theta}$$

تمرین: انرژی جنبشی در مختصات کروی را به‌دست آورید.