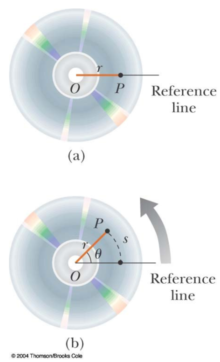
Rotation of Rigid Object

- A rigid object is one in which the relative positions of all the parts is fixed
- What happens when we rotate this object?
  - All points move in a circle about the axis
  - How far do they move?
    $s = r\theta$
  - Note that θ is unitless (ratio of distances) but "measured" in radians

$$\theta = \frac{s}{r}$$



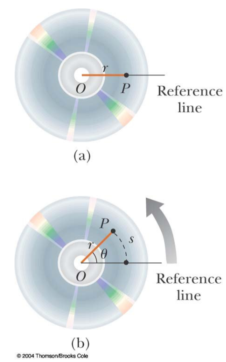
Rotation of Rigid Object

- What if the object is "spinning"
  - It turns thru a given angle every second
  - Define the "angular velocity"

$$\omega = \frac{d\theta}{dt}$$

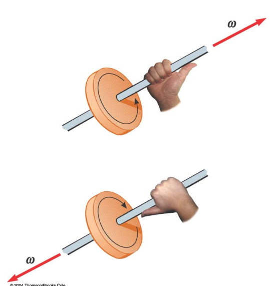
  - Angular velocity measured in radians per second (rad/s or s^{-1})
- What if it's "spinning up"
  - Angular acceleration

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$



Rotation of Rigid Object

- While θ is a scalar, **ω** and **α** are really vectors
- Use "right hand rule" to determine **ω**
- For **α**, it's in the same direction if the magnitude of **ω** is increasing



Linear to Angular and Back

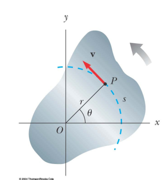
- For each linear quantity (x,v,a) there is a corresponding angular quantity connected by r

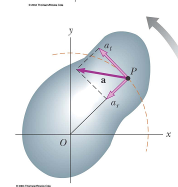
$$r\omega = v$$

$$r\alpha = a$$

- Note that centripetal acceleration is now:

$$a_c = \frac{v^2}{r} = r\omega^2$$





Linear to Angular and Back

- This angular-linear correlation extends to the kinematic equations as well

Table 10.1

Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Rotational Kinetic Energy

- What is the kinetic energy of a spinning object?
  - Just the sum of it's parts!

$$K_R = \sum_i K_i = \tfrac{1}{2}\sum_i m_i v_i^2 = \tfrac{1}{2}\sum_i m_i r_i^2 \omega^2$$

$$= \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I\omega^2$$

$$I = \left(\sum_i m_i r_i^2\right)$$

  - Call I the Moment of Inertia

Rotational Kinetic Energy

- What is the kinetic energy of a rolling object?

$$K_{\text{roll}} = K_R + K_{CM} = \tfrac{1}{2} I\omega^2 + \tfrac{1}{2} Mv^2$$
$$= \tfrac{1}{2}\left(\tfrac{I}{R^2} + M\right)v^2$$

  - Note that for a given energy, a larger I gives a smaller v.
    - Hoops on a ramp go slower than disks

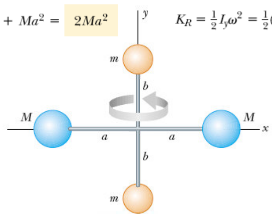
Rotational Kinetic Energy

- What is the kinetic energy of a spinning object?
  - Just the sum of it's parts!

$$K_R = \sum_i K_i = \tfrac{1}{2}\sum_i m_i v_i^2 = \tfrac{1}{2}\sum_i m_i r_i^2 \omega^2$$
$$= \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I\omega^2$$
$$I = \left(\sum_i m_i r_i^2\right)$$

$$I_y = \sum_i m_i r_i^2 = Ma^2 + Ma^2 = \boxed{2Ma^2}$$

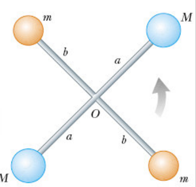
$$K_R = \tfrac{1}{2} I_y\omega^2 = \tfrac{1}{2}(2Ma^2)\omega^2 = \boxed{Ma^2\omega^2}$$



$$I_z = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = \boxed{2Ma^2 + 2mb^2}$$

$$K_R = \tfrac{1}{2} I_z\omega^2 = \tfrac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = \boxed{(Ma^2 + mb^2)\omega^2}$$

If M >> m $\quad I_z = 2Ma^2$ and $K_R = Ma^2\omega^2$



Moment of Inertia

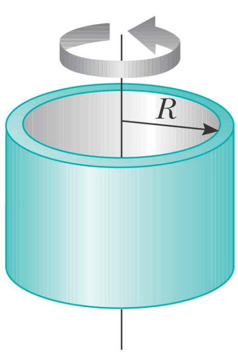
- What is the moment of inertia of an extended object
  - Break it up into little pieces

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i$$
$$= \int_V r^2 dm$$
$$= \int_V r^2 \rho(r) dV$$

Moments of Inertia

- Moment of inertia of a hoop or a thin cylinder
  - All the mass is at the same R!

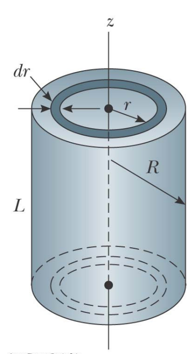
$$I = \int_V r^2 dm$$
$$= R^2 \int_V dm$$
$$= MR^2$$



Moments of Inertia

- Moment of inertia of a disk or solid cylinder
  - Consider ring at r, with volume $2\pi rLdr$

$$I = \int_V r^2 dm = \int_V r^2 \rho dV$$
$$= \rho\int_0^R r^2 (2\pi rLdr) = 2\pi\rho L\int_0^R r^3 dr$$
$$= 2\pi\rho L \tfrac{1}{4} r^4\Big|_0^R = \tfrac{1}{2}\pi\rho LR^4$$
$$= \tfrac{1}{2}\left(\rho\pi R^2 L\right)R^2$$
$$= \tfrac{1}{2} MR^2$$



Moments of Inertia

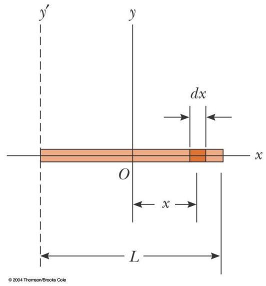
- Moment of inertia of a thin rod about CM

$$I=\int_V r^2 dm=\lambda\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx$$

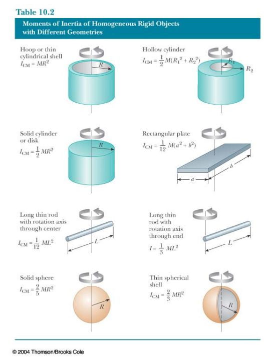
$$=\lambda\tfrac{1}{3}x^3\Big|_{-\frac{L}{2}}^{\frac{L}{2}}=\tfrac{\lambda}{3}\left[\left(\tfrac{L}{2}\right)^3-\left(\tfrac{-L}{2}\right)^3\right]$$

$$=\tfrac{\lambda}{12}L^3=\tfrac{1}{12}(\lambda L)L^2$$

$$=\tfrac{1}{12}ML^2$$



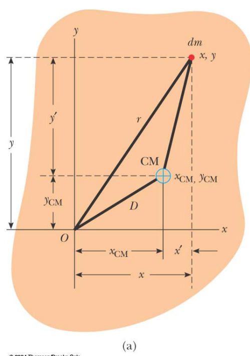
Moments of Inertia



Moments of Inertia

- What about an arbitrary axis?
  - Use "Parallel Axis Theorem"

    $I = I_{CM} + MD^2$
  - Moment of inertia about any axis is just moment of inertia about center of mass plus moment of inertia of "CM" about the axis



(a)

Moments of Inertia

- What about an arbitrary axis?
  - "Parallel Axis Theorem"

$$I=\int r^2 dm=\int\left(x^2+y^2\right)dm$$

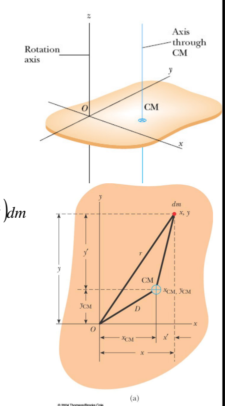
$$=\int\left[\left(x_{CM}+x'\right)^2+\left(y_{CM}+y'\right)^2\right]dm$$

$$=\int\left(x_{CM}{}^2+2x_{CM}x'+x'^2+y_{CM}{}^2+2y_{CM}y'+y'^2\right)dm$$

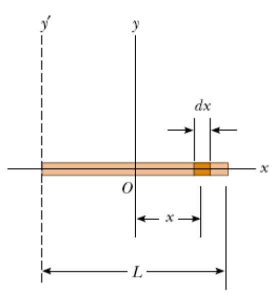
$$=\left(x_{CM}{}^2+y_{CM}{}^2\right)\int dm+\int\left(x'^2+y'^2\right)dm$$

$$+2\int\left(x_{CM}x'+y_{CM}y'\right)dm$$

$$=MD^2+I_{CM}$$



(a)

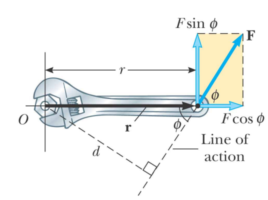


$$I=I_{CM}+MD^2=\tfrac{1}{12}ML^2+M\left(\frac{L}{2}\right)^2=\tfrac{1}{3}ML^2$$

Torque

- If angular acceleration (α) is analogous to acceleration (a), what is analogous to force (F)?
  - Since $\alpha = ra_c$, use

    $\tau = rF_c = rF\sin\phi$
  - Call it "Torque" (like α it too is a vector)

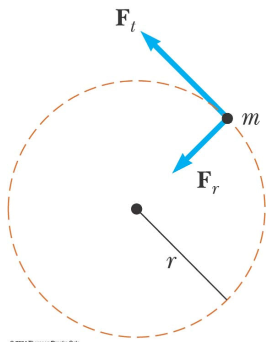


Torque and Angular Acceleration

- Consider a particle at position r
  - What are the kimematics?

$$\tau = rF_t = rma_t$$
$$= rm(r\alpha)$$
$$= mr^2\alpha$$
$$= I\alpha$$

- Analogous to $F=ma$ (linear) we have $\tau=I\alpha$ (angular)

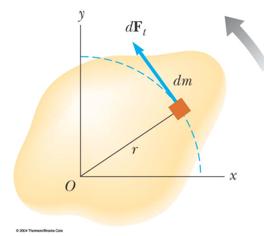


Torque and Angular Acceleration

- What about for extended object?

$$\sum_i \tau_i = \sum_i m r_i^2 \alpha_i$$
$$= \alpha \sum_i m r_i^2$$
$$= I\alpha$$

  - *Net* torque gives rise to angular acceleration



Work and Power

- Work and power for a rotating object

$$dW = Fdx = Fds$$
$$= F\sin\phi \cdot rd\theta$$
$$= \tau d\theta$$
$$P = \frac{dW}{dt} = \frac{\tau d\theta}{dt} = \tau\omega$$
$$W = \int_a^b Pdt = \int_a^b \tau\omega dt = \int_a^b I\alpha\omega dt$$
$$= I\int_a^b \omega \tfrac{d\omega}{dt} dt = \tfrac{1}{2} I \int_a^b \tfrac{d\omega^2}{dt} dt$$
$$= \tfrac{1}{2} I \left(\omega_b^2 - \omega_a^2\right)$$

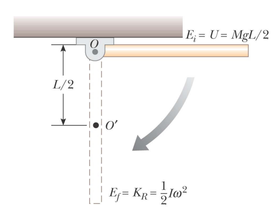
Example: Trapdoor

- What is acceleration of tip when horizontal?

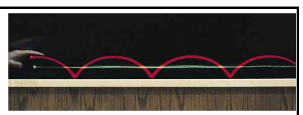
$$\alpha = \frac{\tau}{I} = \left(\frac{3}{ML^2}\right)\left(\frac{MgL}{2}\right) = \frac{3g}{2L}$$
$$a_{\text{tip}} = L\alpha = \tfrac{3}{2} g > g$$

- What is speed of tip when vertical?

$$K_{\text{vert}} = U_{\text{horiz}} = Mg\,\tfrac{L}{2}$$
$$= \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}\left(\tfrac{1}{3} ML^2\right)\omega^2$$
$$\omega = \sqrt{\tfrac{3g}{L}},\ v = L\omega = \sqrt{3gL}$$

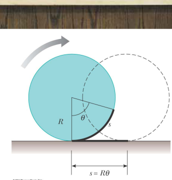


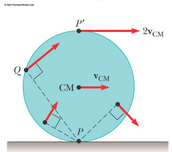
Rolling Motion

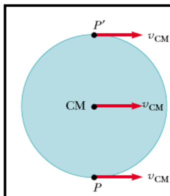


- For a rolling object the point in contact with the surface in momentarily stationary
  - Thus, center (also CM) moves thru a distance $s=r\theta$, at a velocity $v=r\omega$
  - Top moves at speed $2v_{CM}$
- Kinetic energy is rotation plus translation

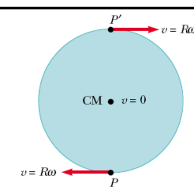
$$K = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M v_{CM}^2$$



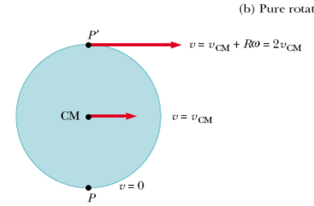




(a) Pure translation



(b) Pure rotation



(c) Combination of translation and rotation

T, a = ?

$$\sum \tau = I\alpha = TR$$

$$\alpha = \frac{TR}{I}$$

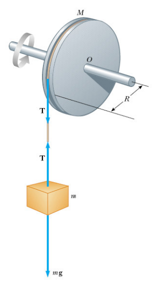
$$\sum F_y = mg - T = ma$$

$$a = \frac{mg - T}{m}$$

$$a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m}$$

$$T = \boxed{\frac{mg}{1 + (mR^2/I)}}$$

$$a = \boxed{\frac{g}{1 + (I/mR^2)}}$$



$$M \xrightarrow{?} \infty$$

$$I \xrightarrow{?} \infty$$

$$a = \frac{g}{1 + (I/mR^2)} \longrightarrow 0$$

$$T = \frac{mg}{1 + (mR^2/I)} \longrightarrow \frac{mg}{1 + 0} = mg$$

$$m_1 g - T_1 = m_1 a$$

$$T_3 - m_2 g = m_2 a$$

$$(T_1 - T_2)R = I\alpha$$

$$(T_2 - T_3)R = I\alpha$$

$$(T_1 - T_3)R = 2I\alpha$$

$$T_3 - T_1 + m_1 g - m_2 g = (m_1 + m_2)a$$

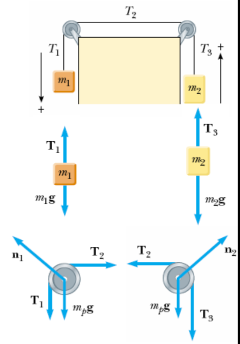
$$T_1 - T_3 = (m_1 - m_2)g - (m_1 + m_2)a$$

$$[(m_1 - m_2)g - (m_1 + m_2)a]R = 2I\alpha$$

$$\alpha = a/R$$

$$(m_1 - m_2)g - (m_1 + m_2)a = 2I\frac{a}{R^2}$$

$$a = \boxed{\frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)}}$$



$$T_1 = m_1 g - m_1 a = m_1(g - a)$$

$$= m_1\left(g - \frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)}\right)$$

$$= 2m_1 g\left(\frac{m_2 + (I/R^2)}{m_1 + m_2 + 2(I/R^2)}\right)$$

$$T_3 = m_2 g + m_2 a = 2m_2 g\left(\frac{m_1 + (I/R^2)}{m_1 + m_2 + 2(I/R^2)}\right)$$

$$T_2 = T_1 - \frac{I\alpha}{R} = T_1 - \frac{Ia}{R^2}$$

$$= 2m_1 g\left(\frac{m_2 + (I/R^2)}{m_1 + m_2 + 2(I/R^2)}\right) - \frac{I}{R^2}\left(\frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)}\right)$$

$$= \frac{2m_1 m_2 + (m_1 + m_2)(I/R^2)}{m_1 + m_2 + 2(I/R^2)}\,g$$

$$M \xrightarrow{?} 0$$

$$I \xrightarrow{?} 0$$

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)} \longrightarrow a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$$

$$T_1 = T_2 = T_3 = T$$

$$T = \left(\frac{2m_1 m_2}{m_1 + m_2}\right)g$$

$$K = \tfrac{1}{2}I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \tfrac{1}{2}Mv_{CM}{}^2$$

$$K = \tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}{}^2$$

$$K_f + U_f = K_i + U_i$$

$$\tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}{}^2 + 0 = 0 + Mgh$$

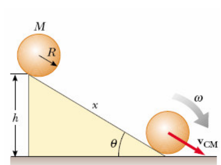
$$v_{CM} = \left(\frac{2gh}{1 + (I_{CM}/MR^2)}\right)^{1/2}$$

$$I_{CM} = \tfrac{2}{5}MR^2$$

$$v_{CM} = \left(\frac{2gh}{1 + (\frac{2}{5}MR^2/MR^2)}\right)^{1/2} = \boxed{(\tfrac{10}{7}gh)^{1/2}}$$

$$v_{CM}{}^2 = \tfrac{10}{7}gx\sin\theta$$

$$v_{CM}{}^2 = 2a_{CM}x \qquad a_{CM} = \tfrac{5}{7}g\sin\theta$$



$$K_f + U_f = K_i + U_i$$

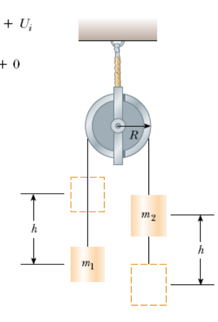
$$(\tfrac{1}{2}m_1 v_f{}^2 + \tfrac{1}{2}m_2 v_f{}^2 + \tfrac{1}{2}I\omega_f{}^2) + (m_1 gh - m_2 gh) = 0 + 0$$

$$\left(\tfrac{1}{2}m_1 v_f{}^2 + \tfrac{1}{2}m_2 v_f{}^2 + \tfrac{1}{2}\frac{I}{R^2}v_f{}^2\right) = (m_2 gh - m_1 gh)$$

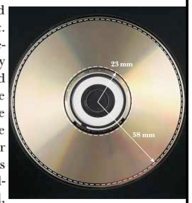
$$\tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f{}^2 = (m_2 gh - m_1 gh)$$

$$v_f = \boxed{\left[\frac{2(m_2 - m_1)gh}{[m_1 + m_2 + (I/R^2)]}\right]^{1/2}}$$

$$\omega_f = \frac{v_f}{R} = \boxed{\frac{1}{R}\left[\frac{2(m_2 - m_1)gh}{(m_1 + m_2 + (I/R^2))}\right]^{1/2}}$$



On a compact disc (Fig. 10.6), audio information is stored in a series of pits and flat areas on the surface of the disc. The information is stored digitally, and the alternations between pits and flat areas on the surface represent binary ones and zeroes to be read by the compact disc player and converted back to sound waves. The pits and flat areas are detected by a system consisting of a laser and lenses. The length of a string of ones and zeroes representing one piece of information is the same everywhere on the disc, whether the information is near the center of the disc or near its outer edge. In order that this length of ones and zeroes always passes by the laser–lens system in the same time period, the tangential speed of the disc surface at the location of the lens must be constant. This requires, according to Equation 10.10, that the angular speed vary as the laser–lens system moves radially along the disc. In a typical compact disc player, the constant speed of the surface at the point of the laser–lens system is 1.3 m/s.



(A) Find the angular speed of the disc in revolutions per minute when information is being read from the innermost first track ($r = 23$ mm) and the outermost final track ($r = 58$ mm).

$$\omega_i = \frac{v}{r_i} = \frac{1.3\ \text{m/s}}{2.3 \times 10^{-2}\ \text{m}} = 57\ \text{rad/s}$$

$$= (57\ \text{rad/s})\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right)$$

$$= \boxed{5.4 \times 10^2\ \text{rev/min}}$$

For the outer track,

$$\omega_f = \frac{v}{r_f} = \frac{1.3\ \text{m/s}}{5.8 \times 10^{-2}\ \text{m}} = 22\ \text{rad/s}$$

$$= \boxed{2.1 \times 10^2\ \text{rev/min}}$$

The player adjusts the angular speed ω of the disc within this range so that information moves past the objective lens at a constant rate.

(B) The maximum playing time of a standard music CD is 74 min and 33 s. How many revolutions does the disc make during that time?

Solution We know that the angular speed is always decreasing, and we assume that it is decreasing steadily, with α constant. If $t = 0$ is the instant that the disc begins, with angular speed of 57 rad/s, then the final value of the time t is $(74\ \text{min})(60\ \text{s/min}) + 33\ \text{s} = 4\,473\ \text{s}$. We are looking for the angular displacement $\Delta\theta$ during this time interval. We use Equation 10.9:

$$\Delta\theta = \theta_f - \theta_i = \tfrac{1}{2}(\omega_i + \omega_f)t$$

$$= \tfrac{1}{2}(57\ \text{rad/s} + 22\ \text{rad/s})(4\,473\ \text{s})$$

$$= 1.8 \times 10^5\ \text{rad}$$

We convert this angular displacement to revolutions:

$$\Delta\theta = 1.8 \times 10^5\ \text{rad}\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = \boxed{2.8 \times 10^4\ \text{rev}}$$

(C) What total length of track moves past the objective lens during this time?

Solution Because we know the (constant) linear velocity and the time interval, this is a straightforward calculation:

$$x_f = v_i t = (1.3\ \text{m/s})(4\,473\ \text{s}) = \boxed{5.8 \times 10^3\ \text{m}}$$

More than 5.8 km of track spins past the objective lens!

(D) What is the angular acceleration of the CD over the 4 473-s time interval? Assume that α is constant.

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{22\ \text{rad/s} - 57\ \text{rad/s}}{4\,473\ \text{s}}$$

$$= \boxed{-7.8 \times 10^{-3}\ \text{rad/s}^2}$$

The disc experiences a very gradual decrease in its rotation rate, as expected.