حرکت دایره ای و کاربردهای دیگر قوانین نیوتن

قبلا شتاب جانب مرکز وارد بر یک جسم در حال حرکت یکنواخت روی یک دایره را به صورت زیر مورد مطالعه قرار دادیم

$$a_r = -\frac{v^2}{r}$$

این مطالعه به طور سینماتیکی (بدون توجه به نیروی پدید آورنده حرکت دایره ای) انجام شد

حال می خواهیم مساله را به طور دینامیکی مورد تحلیل قرار دهیم

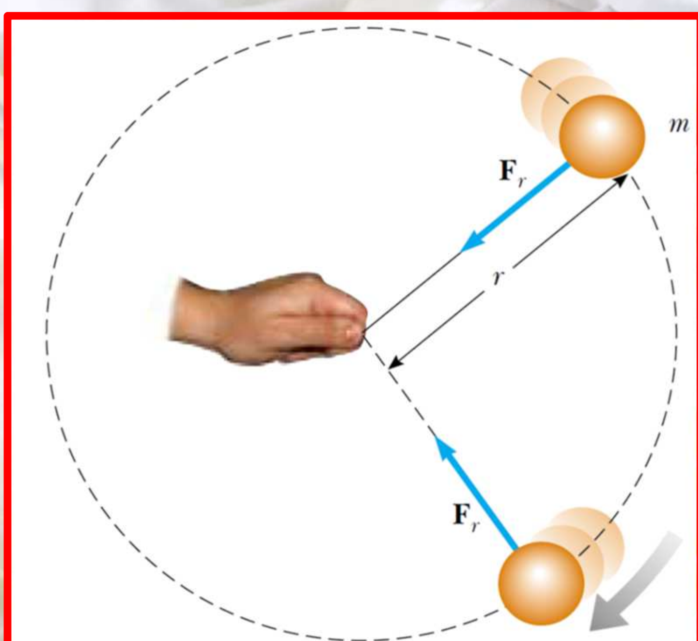
برای این منظور به مطالعه نیرویی که سبب می شود تا جسم روی یک مسیر دایره ای حرکت یکنواخت انجام دهد می پردازیم

چون بنابر قانون دوم نیوتن داریم که:

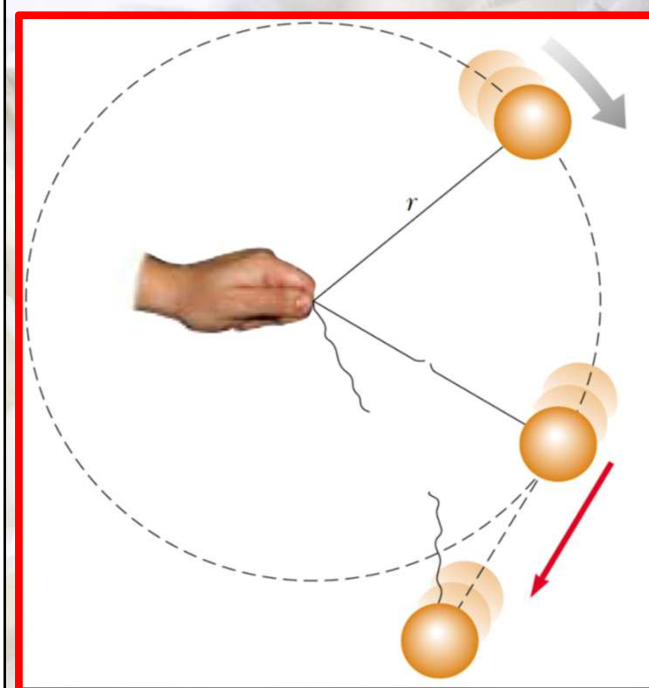
$$F = ma$$

پس خواهیم داشت که:

$$F_r = ma_r = -m\frac{v^2}{r}$$



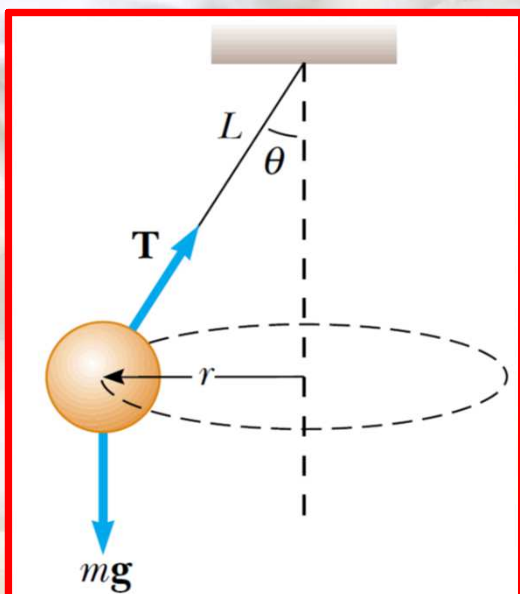
$$F_r = ma_r = -m\frac{v^2}{r}$$



اگر نخ بریده شود، جسم در چه سویی حرکت خواهد کرد؟

آیا جسم در راستای شعاعی از مرکز دور می شود؟

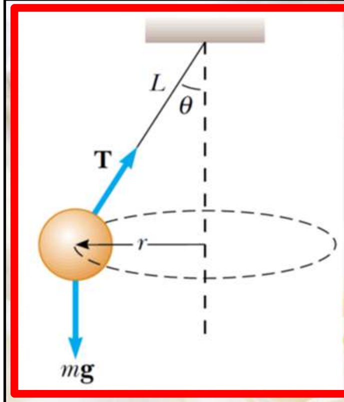
خیر، پس از قطع نخ جسم در راستای مماس بر مسیر دایره ای حرکت خواهد کرد

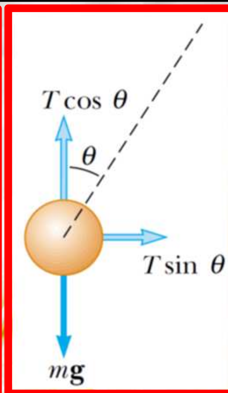


مثال: پاندول مخروطی

می دانیم که هرچه سرعت حرکت دورانی جسم بیشتر شود زاویه هم بزرگتر خواهد شد

سرعت جسم را بر حسب زاویه به دست آورید





$$T\cos\theta = mg$$

$$T\sin\theta = \frac{mv^2}{r}$$

$$\frac{T\sin\theta}{T\cos\theta} = \frac{\frac{mv^2}{r}}{mg}$$

$$\tan\theta = \frac{v^2}{rg}$$

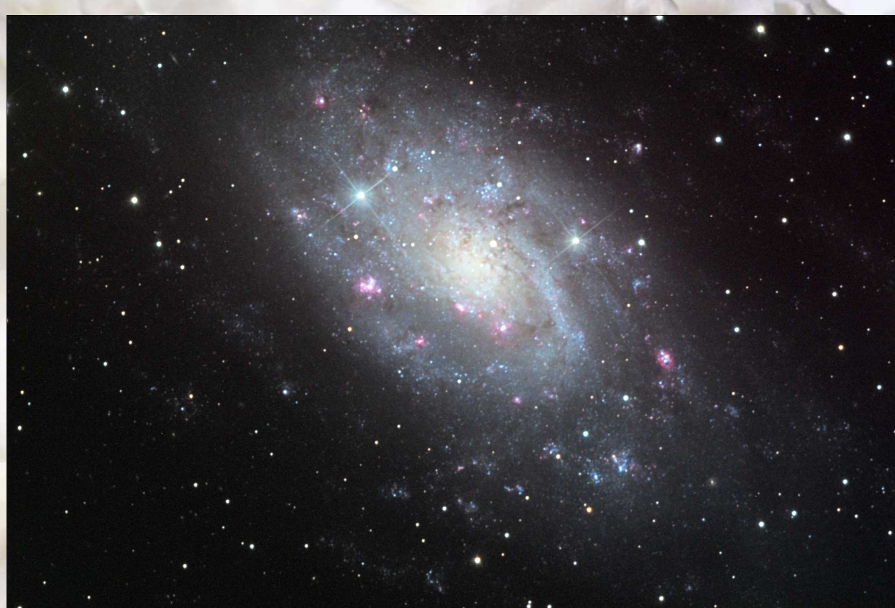
$$v^2 = gr\tan\theta$$

$$\sin\theta = r/L$$

$$L\sin\theta = r$$

$$v = \sqrt{gL\sin\theta\tan\theta}$$

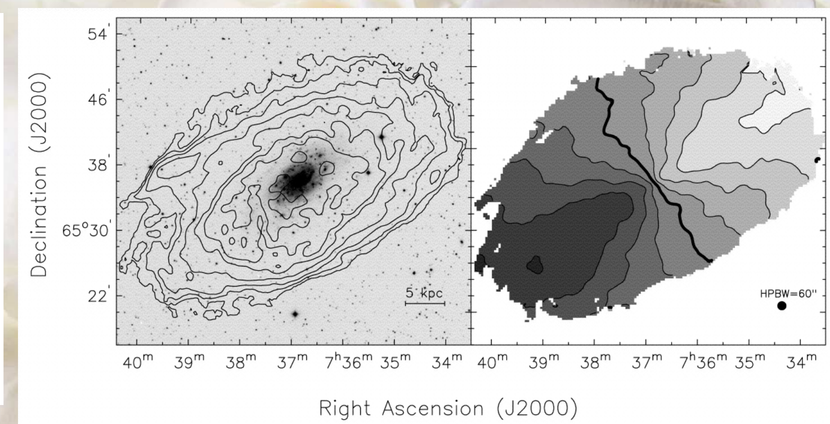
مثال: دورانهای کهکشانی



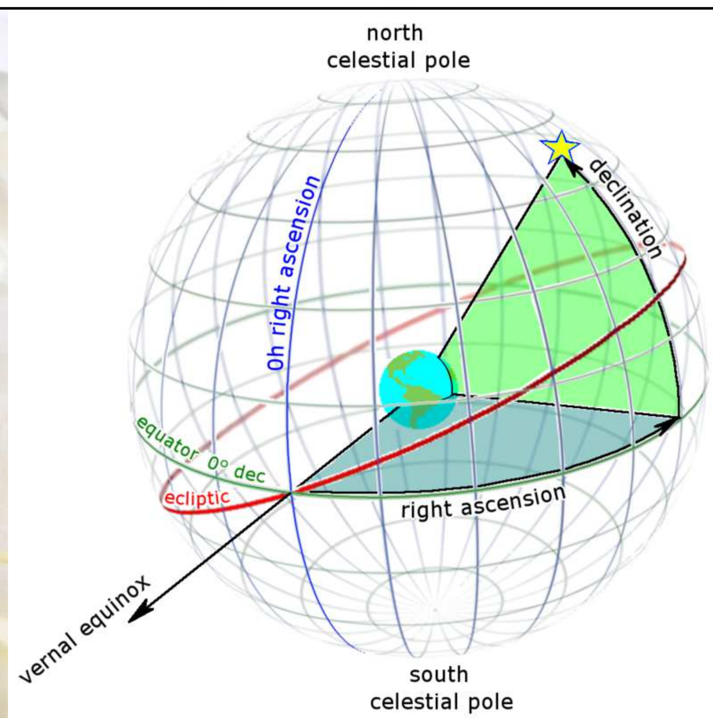
ستاره ها در کهکشانهای خود در دوایری با سرعتهای متفاوت در حال چرخش هستند

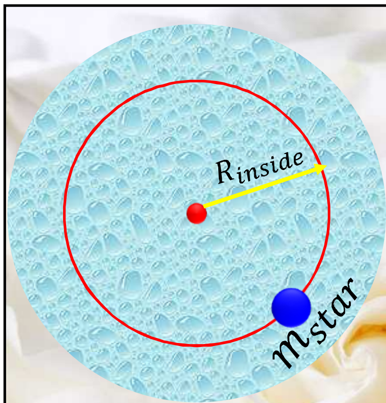
سرعت ستاره ها با استفاده از پدیده دوپلر نوری و تغییر فرکانس آنها (هنگام دور شدن از یا نزدیک شدن به ما) به طور تجربی اندازه گیری می شود

1 kpc = 3.08568025 × 10^{19} meters



Half-Power Beam Width





$$v_{star}^2 = G\frac{M_{inside}}{R_{inside}}$$

$$v_{star} = \sqrt{\frac{GM_{inside}}{R_{inside}}}$$

$$F = G\frac{m_{star}M_{inside}}{R_{inside}^2}$$

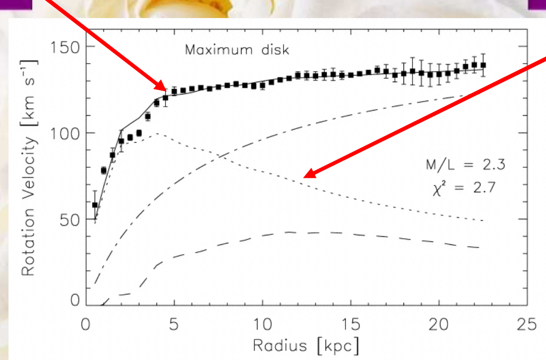
$$\frac{m_{star}v_{star}^2}{R_{inside}} = G\frac{m_{star}M_{inside}}{R_{inside}^2}$$

تفاوت بین نتایج تجربی و نظری
ناشی از وجود سیاه چاله هاست

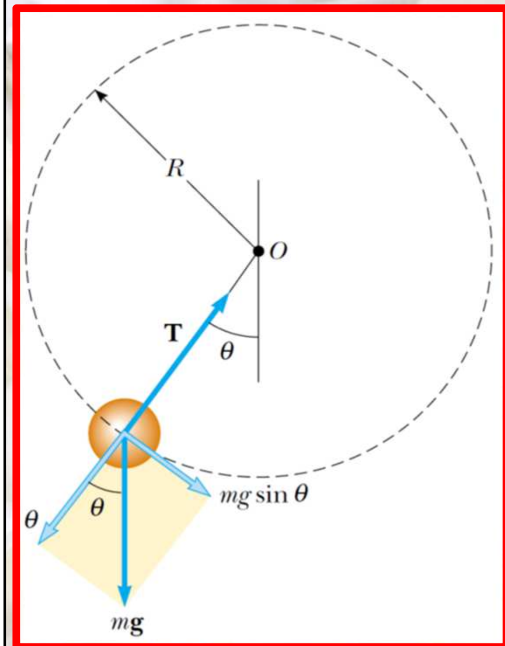
$$v_{inside} = \sqrt{\frac{GM_{inside}}{R_{inside}}}$$

منحنی تجربی

منحنی نظری



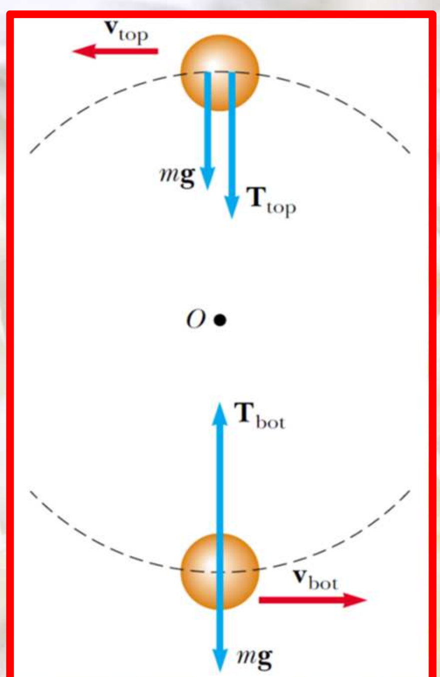
حرکت دایره ای غیر یکنواخت



حرکت دایره ای یک جسم متصل به نخ به طور عمودی

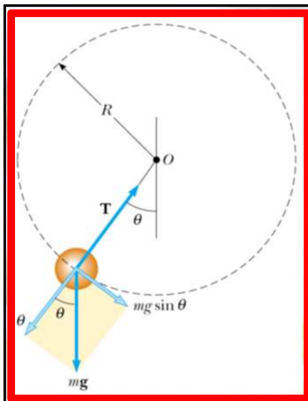
در این صورت دیگر سرعت ثابت نخواهد ماند

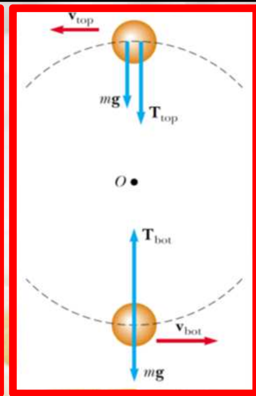
سرعت جسم و کشش نخ توابعی از زاویه می باشند



کشش نخ در بالاترین نقطه کمترین مقدار را دارد

کشش نخ در پایینترترین نقطه بیشترین مقدار را دارد





$$T = m\left(\frac{v^2}{R} + g\cos\theta\right)$$

نیروی کشش نخ در بالاترین نقطه:

$$T = m\left(\frac{v^2}{R} - g\right)$$

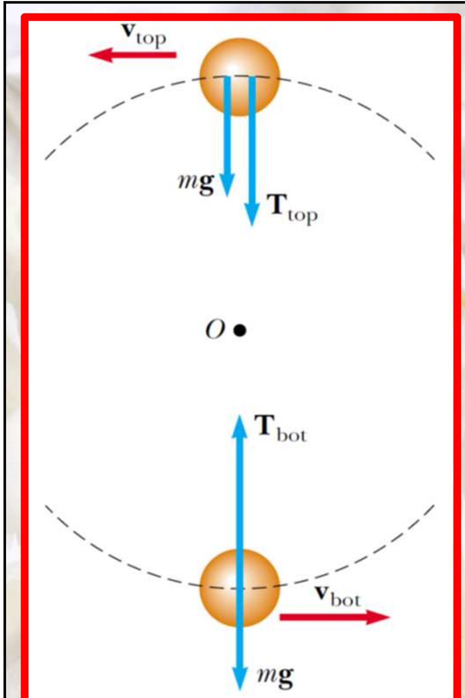
نیروی کشش نخ در پایینترین نقطه:

$$T = m\left(\frac{v^2}{R} + g\right)$$

$$ma_\theta = m\frac{dv_\theta}{dt} = m\frac{dv}{dt} = mg\sin\theta$$

$$a_\theta = g\sin\theta$$

$$ma_r = \frac{mv^2}{R} = T - mg\cos\theta$$



کمترین سرعتی که می تواند جسم با آن به بالاترین نقطه برسد

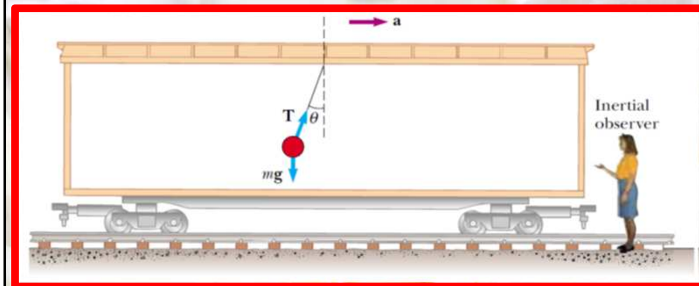
$$T = m\left(\frac{v^2}{R} - g\right) = 0$$

$$\frac{v^2}{R} - g = 0$$

$$v^2 = Rg$$

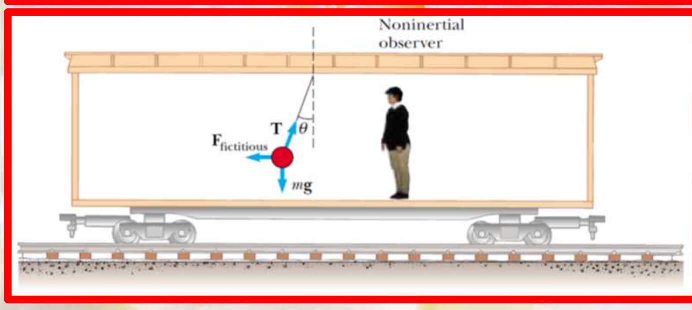
$$v = \sqrt{Rg}$$

دستگاه های غیر لخت



مشاهده گر لخت می گوید که جسم شتاب دارد و مولفه افقی نیروی کشش نخ سبب شتاب جسم شده است

$$T\sin\theta = ma$$

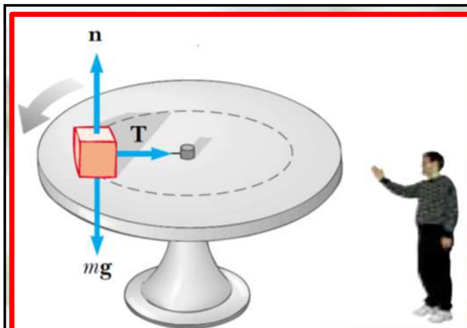


مشاهده گر غیرلخت می گوید که جسم ساکن است و مولفه افقی نیروی کشش نخ توسط نیروی مجازی وارده (که خود شخص نیز آن را حس می کند) آن را خنثی کرده است

$$T\sin\theta = F_{fic}$$

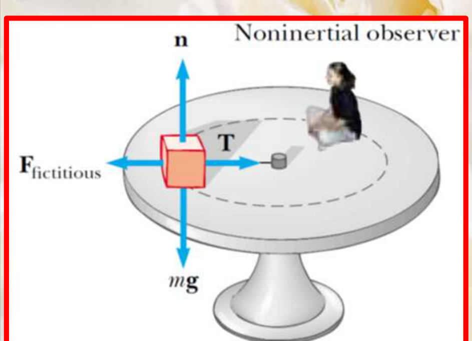
$$F_{fic} = ma$$

که نتیجه هر دو سازگار با یکدیگر است



مشاهده گر لخت می گوید که نیروی کشش نخ به جسم نیروی جانب مرکز وارد می کند و سبب حرکت دایروی آن می شود

$$T = \frac{mv^2}{R}$$



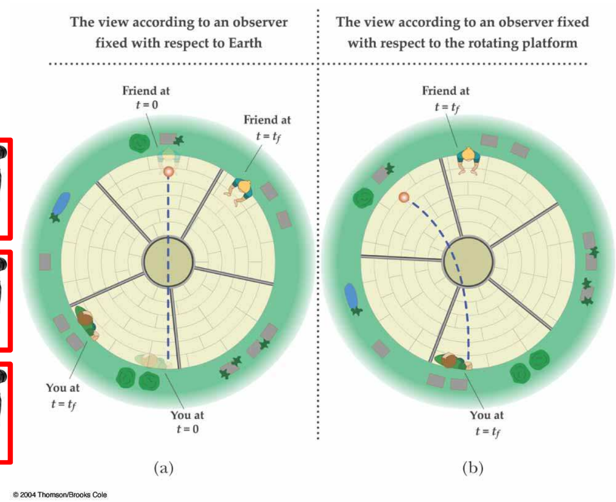
مشاهده گر غیرلخت می گوید که نیروی کشش نخ به وسیله نیروی مجازی گریز از مرکز خنثی می شود

$$T - F_{fic} = 0$$

$$F_{fic} = \frac{mv^2}{R}$$

که نتیجه هر دو سازگار با یکدیگر است

نیروی کوریولی



دستگاه 2 بعدی و دارای تقارن قطبی (دایروی) است

حرکت دایره ای یکنواخت

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

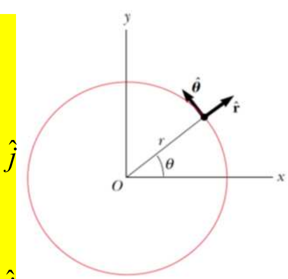
$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$

$$\frac{d\hat{\theta}}{d\theta} = -\hat{r}$$



$$\mathbf{r} = R\hat{\mathbf{r}}$$

$$\cancel{\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dR}{dt}\hat{\mathbf{r}} = 0\hat{\mathbf{r}}}$$

$$\frac{d\mathbf{r}}{dt} = \cancel{\frac{dR}{dt}}\hat{\mathbf{r}} + R\frac{d\hat{\mathbf{r}}}{dt} = R\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = v\hat{\boldsymbol{\theta}} = \mathbf{v}$$

a_r

$$\frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\hat{\boldsymbol{\theta}} + v\frac{d\hat{\boldsymbol{\theta}}}{dt} = a_\theta\hat{\boldsymbol{\theta}} + v\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = a_\theta\hat{\boldsymbol{\theta}} + v\frac{d\theta}{dt}\left(-\hat{\mathbf{r}}\right) = a_\theta\hat{\boldsymbol{\theta}} - \frac{v^2}{R}\hat{\mathbf{r}}$$

حرکت دایره ای غیریکنواخت

$$\mathbf{r} = r\hat{\mathbf{r}}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d(r\hat{\mathbf{r}})}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}})}{dt} = \frac{d(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})}{dt} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} +$$

$$+ r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta}\,\hat{\mathbf{r}})$$

$$\mathbf{a} = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{r}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}$$

Coriolis Acceleration

نیروی میرای وابسته به سرعت

در اینجا فرض می کنیم که F فقط تابع صریحی از سرعت باشد:

$$F = f(v)$$

$$m\frac{dv}{dt} = F(v)$$

$$\frac{dv}{F(v)} = \frac{dt}{m}$$

$$\int_{v_0}^{v}\frac{dv}{F(v)} = \int_{t_0}^{t}\frac{dt}{m}$$

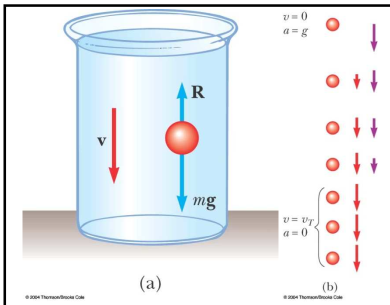
$$\int_{v_0}^{v}\frac{dv}{F(v)} = \frac{t-t_0}{m}$$

$$f(v, v_0) = \frac{t-t_0}{m}$$

$$v = \frac{dx}{dt} = \varphi\left(v_0, \frac{t-t_0}{m}\right)$$

$$dx = \varphi\left(v_0, \frac{t-t_0}{m}\right)dt$$

$$x = x_0 + \int_{t_0}^{t}\varphi\left(v_0, \frac{t-t_0}{m}\right)dt$$



یکی از مثالهایی که در آن F تابع سرعت است، نیروی اصطحکاک است

نیروی اصطحکاک همواره در خلاف جهت سرعت است

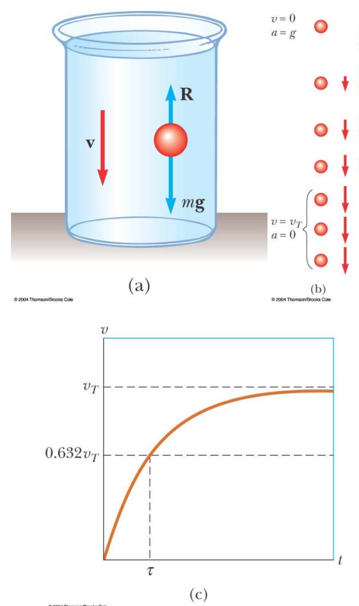
نیروی اصطحکاک در حالت کلی می تواند تابع پیچیده ای از سرعت باشد

اما در بسیاری از موارد شکل ساده زیر را دارد

$$F = \begin{cases} -bv^n & n = 2k+1 \quad for \pm v \\ -bv^n & n = 2k \quad for + v \\ bv^n & n = 2k \quad for - v \end{cases}$$

در اینجا حالت n=1 را بررسی می کنیم

اجسام افتان (در حضور اصطحکاک)به عنوان مثالی از نیروی وابسته به سرعت



$$ma = -mg - bv = 0$$

$$v_T = -\frac{mg}{b}$$

$$\frac{dv}{dt} = -g - \frac{b}{m}v$$

$$v = -\frac{mg}{b}\left(1 - e^{-bt/m}\right) = v_T\left(1 - e^{-t/\tau}\right)$$

$$F = ma$$

$$-bv - mg = ma$$

$$-\frac{b}{m}v - g = a$$

$$-\frac{b}{m}v - g = \frac{dv}{dt}$$

$$\left(\frac{b}{m}v + g\right)dt = -dv$$

$$\frac{dv}{\left(\frac{b}{m}v + g\right)} = -dt$$

$$\int_{v_0}^{v} \frac{dv}{\left(\frac{b}{m}v + g\right)} = -\int_0^t dt$$

$$\int_{v_0}^{v} \frac{dv}{\left(\frac{b}{m}v + g\right)} = -\int_0^t dt$$

$$\frac{b}{m}v + g = u$$

$$\frac{b}{m}dv = du$$

$$\int \frac{du}{u^n} = \begin{cases} \dfrac{u^{-n+1}}{-n+1} & n \neq 1 \\ \ln u & n = 1 \end{cases}$$

$$\int \frac{\frac{m}{b}du}{u} = -\int_0^t dt$$

$$\frac{m}{b}\int \frac{du}{u} = -\int_0^t dt$$

$$\frac{m}{b}\ln u = -t$$

$$\frac{m}{b}\ln\left(\frac{b}{m}v + g\right)\Bigg|_{v_o}^{v} = -t$$

$$\frac{m}{b}\left\{\ln\left(\frac{b}{m}v+g\right)-\ln\left(\frac{b}{m}v_0+g\right)\right\}=-t$$

$$\frac{m}{b}\ln\left|\frac{\frac{b}{m}v+g}{\frac{b}{m}v_0+g}\right|=-t$$

$$\ln\left|\frac{\frac{b}{m}v+g}{\frac{b}{m}v_0+g}\right|=-\frac{b}{m}t$$

$$\ln a=b \equiv \log_e a=b$$

$$a=e^b$$

$$\frac{\frac{b}{m}v+g}{\frac{b}{m}v_0+g}=e^{-\frac{b}{m}t}=\exp\left\{-\frac{b}{m}t\right\}$$

$$\frac{b}{m}v+g=(\frac{b}{m}v_0+g)e^{-\frac{b}{m}t}$$

$$v+\frac{m}{b}g=v_0e^{-\frac{b}{m}t}+\frac{m}{b}ge^{-\frac{b}{m}t}$$

$$v=v_0e^{-\frac{b}{m}t}-\frac{m}{b}g+\frac{m}{b}ge^{-\frac{b}{m}t}$$

$$v=v_0e^{-\frac{b}{m}t}-\frac{m}{b}g\left(1-e^{-\frac{b}{m}t}\right)$$

$$v=-\frac{mg}{b}\left(1-e^{-\frac{b}{m}t}\right)$$

$$v=\tau g\left(1-e^{-\frac{t}{\tau}}\right)$$

$$v=v_T\left(1-e^{-\frac{t}{\tau}}\right)$$

$$v=-\frac{m}{b}g\left(1-e^{-\frac{b}{m}t}\right)$$

$$v=-\frac{m}{b}g\left(1-\left(1-\frac{b}{m}t+\frac{1}{2}\left(\frac{b}{m}\right)^2t^2\right)+\ldots\right)$$

$$v\cong -gt+\frac{1}{2}\frac{bg}{m}t^2$$

$$t<<m/b\Rightarrow v\cong -gt$$

یک قایق که با سرعت اولیه v_0 در حال حرکت است موتور خود را خاموش می کند. سرعت آن را در حضور اصطحکاک متناسب با سرعت به صورت تابعی از زمان به دست آورید.

$$m\frac{dv}{dt} = -bv$$

$$\frac{dv}{v} = -\frac{b}{m}dt$$

$$\int_{v_0}^{v}\frac{dv}{v} = -\frac{b}{m}\int_0^t dt$$

$$lnv\Big|_{v_0}^{v} = -\frac{b}{m}t$$

$$ln\frac{v}{v_0} = -\frac{b}{m}t$$

$$\frac{v}{v_0} = exp\left(-\frac{b}{m}t\right)$$

$$v = v_0 exp\left(-\frac{b}{m}t\right)$$

$$\lim_{t\to\infty} v = 0$$

تقریب بهتری برای اصطحکاک در اجسام سنگین و کوچک با سرعتهای نهایی بزرگ به صورت زیر در نظر گرفته می شود:

$$F = bv^2$$

$$b = \frac{1}{2}DA\rho^2$$

ρ چگالی

A سطح مقطع

D ضریب کشش

$D=0.5$ برای کره

Table 6.1

Terminal Speed for Various Objects Falling Through Air

Object	Mass (kg)	Cross-Sectional Area (m^2)	v_T (m/s)
Sky diver	75	0.70	60
Baseball (radius 3.7 cm)	0.145	4.2×10^{-3}	43
Golf ball (radius 2.1 cm)	0.046	1.4×10^{-3}	44
Hailstone (radius 0.50 cm)	4.8×10^{-4}	7.9×10^{-5}	14
Raindrop (radius 0.20 cm)	3.4×10^{-5}	1.3×10^{-5}	9.0

$$ma = -mg + bv^2 = 0$$

$$v_T = -\sqrt{\frac{mg}{b}} = -\sqrt{\frac{2mg}{D\rho A}}$$

$$F = ma$$

$$+bv^2 - mg = ma$$

$$+\frac{b}{m}v^2 - g = a$$

$$+\frac{b}{m}v^2 - g = \frac{dv}{dt}$$

$$\left(-\frac{b}{m}v^2 + g\right)dt = -dv$$

$$\frac{dv}{\left(-\frac{b}{m}v^2 + g\right)} = -dt$$

$$\int_0^v \frac{dv}{\left(-\frac{b}{m}v^2 + g\right)} = -\int_0^t dt$$

$$\int_0^v \frac{dv}{g\left(1 - \left(\sqrt{\frac{b}{mg}}v\right)^2\right)} = -t$$

$$\sqrt{\frac{b}{mg}}v = \tanh u$$

$$\sqrt{\frac{b}{mg}}v = \tanh u$$

$$\sqrt{\frac{b}{mg}}dv = \frac{1}{\cosh^2 u}du$$

$$\int \frac{\sqrt{\frac{mg}{b}}\sec h^2 u du}{g\left(1 - \tanh^2 u\right)} = -t$$

$$\int \frac{\sec h^2 u du}{\sec h^2 u} = -\sqrt{\frac{bg}{m}}t$$

$$\int du = -\sqrt{\frac{bg}{m}}t$$

$$u = -\sqrt{\frac{bg}{m}}t$$

$$\tanh^{-1}\left(\sqrt{\frac{b}{mg}}v\right) = -\sqrt{\frac{bg}{m}}t$$

$$\sqrt{\frac{b}{mg}}v = \tanh\left(-\sqrt{\frac{bg}{m}}t\right)$$

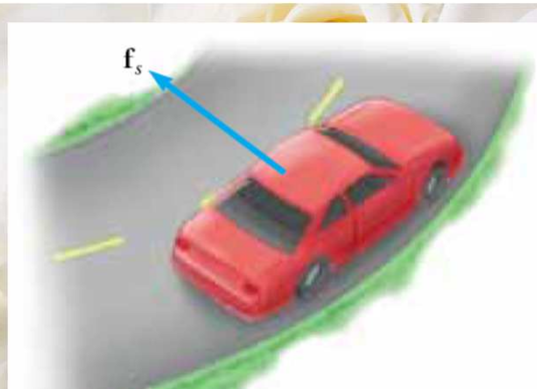
$$v = -\sqrt{\frac{mg}{b}}\tanh\left(\sqrt{\frac{bg}{m}}t\right)$$

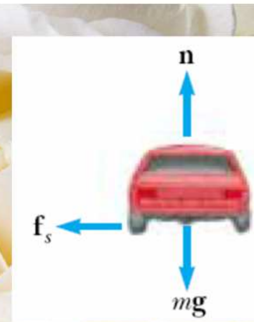
$$\tanh x \approx \begin{cases} x - x^3/3 & \mathrm{x} \ll 1 \\ 1 & \mathrm{x} \gg 1 \end{cases}$$

$$\boxed{v = -\sqrt{\frac{mg}{b}}\tanh\left(\sqrt{\frac{bg}{m}}t\right) = \begin{cases} -gt & t \ll \sqrt{\dfrac{m}{bg}} \\ -\sqrt{\dfrac{mg}{b}} & t \gg \sqrt{\dfrac{m}{bg}} \end{cases}}$$

Example 6.4 What Is the Maximum Speed of the Car?

A 1 500-kg car moving on a flat, horizontal road negotiates a curve, as shown in Figure 6.5. If the radius of the curve is 35.0 m and the coefficient of static friction between the tires and dry pavement is 0.500, find the maximum speed the car can have and still make the turn successfully.





Solution In this case, the force that enables the car to remain in its circular path is the force of static friction. (*Static* because no slipping occurs at the point of contact between road and tires. If this force of static friction were zero—for example, if the car were on an icy road—the car would continue in a straight line and slide off the road.) Hence, from Equation 6.1 we have

$$(1) \qquad f_s = m\frac{v^2}{r}$$

The maximum speed the car can have around the curve is the speed at which it is on the verge of skidding outward. At this point, the friction force has its maximum value $f_{s,\,\max} = \mu_s n$. Because the car shown in Figure 6.5b is in equilibrium in the vertical direction, the magnitude of the normal force equals the weight ($n = mg$) and thus $f_{s,\,\max} = \mu_s mg$. Substituting this value for f_s into (1), we find that the maximum speed is

$$(2) \qquad v_{\max} = \sqrt{\frac{f_{s,\,\max}\, r}{m}} = \sqrt{\frac{\mu_s m g r}{m}} = \sqrt{\mu_s g r}$$

$$= \sqrt{(0.500)(9.80\ \mathrm{m/s^2})(35.0\ \mathrm{m})}$$

$$= \boxed{13.1\ \mathrm{m/s}}$$

Note that the maximum speed does not depend on the mass of the car. That is why curved highways do not need multiple speed limit signs to cover the various masses of vehicles using the road.

What If? Suppose that a car travels this curve on a wet day and begins to skid on the curve when its speed reaches only 8.00 m/s. What can we say about the coefficient of static friction in this case?

Answer The coefficient of friction between tires and a wet road should be smaller than that between tires and a dry road. This expectation is consistent with experience with driving, because a skid is more likely on a wet road than a dry road.

To check our suspicion, we can solve (2) for the coefficient of friction:

$$\mu_s = \frac{v_{\text{max}}^2}{gr}$$

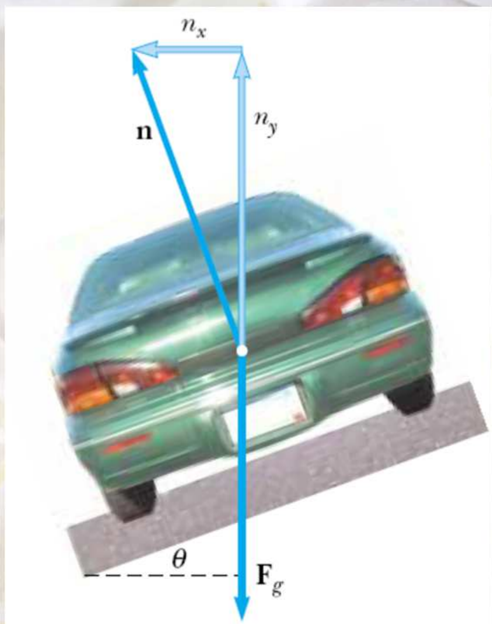
Substituting the numerical values,

$$\mu_s = \frac{v_{\text{max}}^2}{gr} = \frac{(8.00 \text{ m/s})^2}{(9.80 \text{ m/s}^2)(35.0 \text{ m})} = 0.187$$

This is indeed smaller than the coefficient of 0.500 for the dry road.

Example 6.5 The Banked Exit Ramp

A civil engineer wishes to design a curved exit ramp for a highway in such a way that a car will not have to rely on friction to round the curve without skidding. In other words, a car moving at the designated speed can negotiate the curve even when the road is covered with ice. Such a ramp is usually *banked*; this means the roadway is tilted toward the inside of the curve. Suppose the designated speed for the ramp is to be 13.4 m/s (30.0 mi/h) and the radius of the curve is 50.0 m. At what angle should the curve be banked?



Solution On a level (unbanked) road, the force that causes the centripetal acceleration is the force of static friction between car and road, as we saw in the previous example. However, if the road is banked at an angle θ, as in Figure 6.6, the normal force $\mathbf{n}$ has a horizontal component $n \sin\theta$ pointing toward the center of the curve. Because the ramp is to be designed so that the force of static friction is zero, only the component $n_x = n \sin\theta$ causes the centripetal acceleration. Hence, Newton's second law for the radial direction gives

$$(1) \qquad \sum F_r = n \sin\theta = \frac{mv^2}{r}$$

The car is in equilibrium in the vertical direction. Thus, from $\Sigma F_y = 0$ we have

$$(2) \qquad n\cos\theta = mg$$

Dividing (1) by (2) gives

$$(3) \qquad \tan\theta = \frac{v^2}{rg}$$

$$\theta = \tan^{-1}\left(\frac{(13.4\ \text{m/s})^2}{(50.0\ \text{m})(9.80\ \text{m/s}^2)}\right) = 20.1°$$

If a car rounds the curve at a speed less than 13.4 m/s, friction is needed to keep it from sliding down the bank (to the left in Fig. 6.6). A driver who attempts to negotiate the curve at a speed greater than 13.4 m/s has to depend on friction to keep from sliding up the bank (to the right in Fig. 6.6). The banking angle is independent of the mass of the vehicle negotiating the curve.

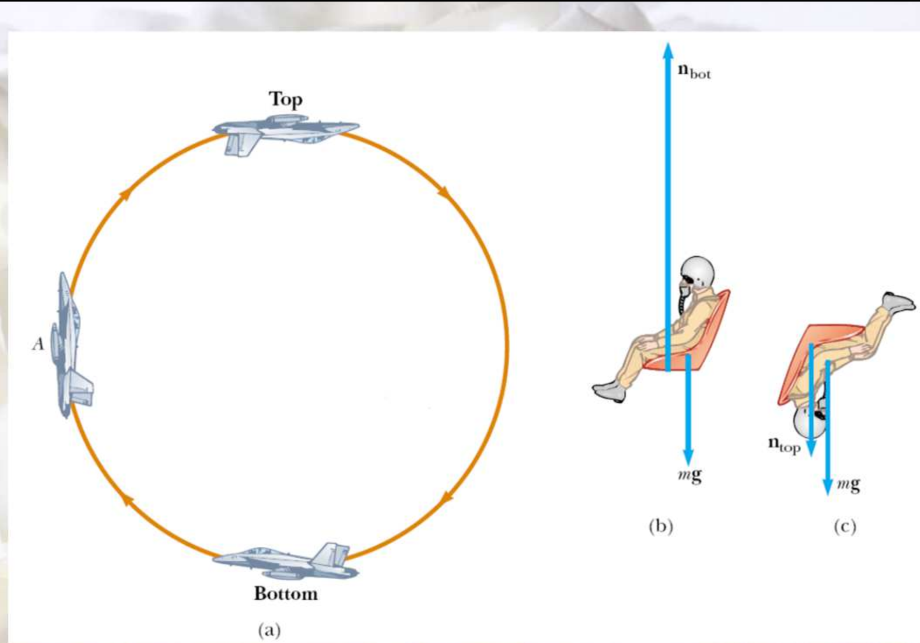
What If? **What if this same roadway were built on Mars in the future to connect different colony centers; could it be traveled at the same speed?**

Answer The reduced gravitational force on Mars would mean that the car is not pressed so tightly to the roadway. The reduced normal force results in a smaller component of the normal force toward the center of the circle. This smaller component will not be sufficient to provide the centripetal acceleration associated with the original speed. The centripetal acceleration must be reduced, which can be done by reducing the speed v.

Equation (3) shows that the speed v is proportional to the square root of g for a roadway of fixed radius r banked at a fixed angle θ. Thus, if g is smaller, as it is on Mars, the speed v with which the roadway can be safely traveled is also smaller.

Example 6.6 Let's Go Loop-the-Loop!

A pilot of mass m in a jet aircraft executes a loop-the-loop, as shown in Figure 6.7a. In this maneuver, the aircraft moves in a vertical circle of radius 2.70 km at a constant speed of 225 m/s. Determine the force exerted by the seat on the pilot **(A)** at the bottom of the loop and **(B)** at the top of the loop. Express your answers in terms of the weight of the pilot mg.



Solution To conceptualize this problem, look carefully at Figure 6.7. Based on experiences with driving over small hills in a roadway, or riding over the top of a Ferris wheel, you would expect to feel lighter at the top of the path. Similarly, you would expect to feel heavier at the bottom of the path. By looking at Figure 6.7, we expect the answer for (A) to be greater than that for (B) because at the bottom of the loop the normal and gravitational forces act in *opposite* directions, whereas at the top of the loop these two forces act in the *same* direction. The vector sum of these two forces gives the force of constant magnitude that keeps the pilot moving in a circular path at a constant speed. To yield net force vectors with the same magnitude, the normal force at the bottom must be greater than that at the top. Because the speed of the aircraft is constant (how likely is this?), we can categorize this as a uniform circular motion problem, complicated by the fact that the gravitational force acts at all times on the aircraft.

(A) Analyze the situation by drawing a free-body diagram for the pilot at the bottom of the loop, as shown in Figure 6.7b. The only forces acting on him are the downward gravitational force $\mathbf{F}_g = m\mathbf{g}$ and the upward force $\mathbf{n}_{\text{bot}}$ exerted by the seat. Because the net upward force that provides the centripetal acceleration has a magnitude $n_{\text{bot}} - mg$, Newton's second law for the radial direction gives

$$\sum F = n_{\text{bot}} - mg = m\frac{v^2}{r}$$

$$n_{\text{bot}} = mg + m\frac{v^2}{r} = mg\left(1 + \frac{v^2}{rg}\right)$$

Substituting the values given for the speed and radius gives

$$n_{\text{bot}} = mg\left(1 + \frac{(225\ \text{m/s})^2}{(2.70 \times 10^3\ \text{m})(9.80\ \text{m/s}^2)}\right) = \boxed{2.91\,mg}$$

Hence, the magnitude of the force $\mathbf{n}_{\text{bot}}$ exerted by the seat on the pilot is *greater* than the weight of the pilot by a factor of 2.91. This means that the pilot experiences an apparent weight that is greater than his true weight by a factor of 2.91.

(B) The free-body diagram for the pilot at the top of the loop is shown in Figure 6.7c. As we noted earlier, both the gravitational force exerted by the Earth and the force $\mathbf{n}_{top}$ exerted by the seat on the pilot act downward, and so the net downward force that provides the centripetal acceleration has a magnitude $n_{top} + mg$. Applying Newton's second law yields

$$\sum F = n_{top} + mg = m\frac{v^2}{r}$$

$$n_{top} = m\frac{v^2}{r} - mg = mg\left(\frac{v^2}{rg} - 1\right)$$

$$n_{top} = mg\left(\frac{(225\text{ m/s})^2}{(2.70 \times 10^3\text{ m})(9.80\text{ m/s}^2)} - 1\right)$$

$$= \boxed{0.913mg}$$

In this case, the magnitude of the force exerted by the seat on the pilot is *less* than his true weight by a factor of 0.913, and the pilot feels lighter. To finalize the problem, note that this is consistent with our prediction at the beginning of the solution.

Example 6.10 Sphere Falling in Oil

A small sphere of mass 2.00 g is released from rest in a large vessel filled with oil, where it experiences a resistive force proportional to its speed. The sphere reaches a terminal speed of 5.00 cm/s. Determine the time constant τ and the time at which the sphere reaches 90.0% of its terminal speed.

Solution Because the terminal speed is given by $v_T = mg/b$, the coefficient b is

$$b = \frac{mg}{v_T} = \frac{(2.00\ \text{g})(980\ \text{cm/s}^2)}{5.00\ \text{cm/s}} = 392\ \text{g/s}$$

Therefore, the time constant τ is

$$\tau = \frac{m}{b} = \frac{2.00\ \text{g}}{392\ \text{g/s}} = 5.10 \times 10^{-3}\ \text{s}$$

The speed of the sphere as a function of time is given by Equation 6.5. To find the time t at which the sphere reaches a speed of $0.900v_T$, we set $v = 0.900v_T$ in Equation 6.5 and solve for t:

$$0.900v_T = v_T(1 - e^{-t/\tau})$$

$$1 - e^{-t/\tau} = 0.900$$

$$e^{-t/\tau} = 0.100$$

$$-\frac{t}{\tau} = \ln(0.100) = -2.30$$

$$t = 2.30\tau = 2.30(5.10 \times 10^{-3}\ \text{s})$$

$$= 11.7 \times 10^{-3}\ \text{s} = 11.7\ \text{ms}$$

Thus, the sphere reaches 90.0% of its terminal speed in a very short time interval.

Quick Quiz 6.7 A baseball and a basketball, having the same mass, are dropped through air from rest such that their bottoms are initially at the same height above the ground, on the order of 1 m or more. Which one strikes the ground first? (a) the baseball (b) the basketball (c) both strike the ground at the same time.

Conceptual Example 6.11 The Sky Surfer

Consider a sky surfer (Fig. 6.17) who jumps from a plane with her feet attached firmly to her surfboard, does some tricks, and then opens her parachute. Describe the forces acting on her during these maneuvers.

Solution When the surfer first steps out of the plane, she has no vertical velocity. The downward gravitational force causes her to accelerate toward the ground. As her downward speed increases, so does the upward resistive force exerted by the air on her body and the board. This upward force reduces their acceleration, and so their speed increases more slowly. Eventually, they are going so fast that the upward resistive force matches the downward gravitational force. Now the net force is zero and they no longer accelerate, but reach their terminal speed. At some point after reaching terminal speed, she opens her parachute, resulting in a drastic increase in the upward resistive force. The net force (and thus the acceleration) is now upward, in the direction opposite the direction of the velocity. This causes the downward velocity to decrease rapidly; this means the resistive force on the chute also decreases. Eventually the upward resistive force and the downward gravitational force balance each other and a much smaller terminal speed is reached, permitting a safe landing.

(Contrary to popular belief, the velocity vector of a sky diver never points upward. You may have seen a videotape in which a sky diver appears to "rocket" upward once the chute opens. In fact, what happens is that the diver slows down while the person holding the camera continues falling at high speed.)

Example 6.12 Falling Coffee Filters

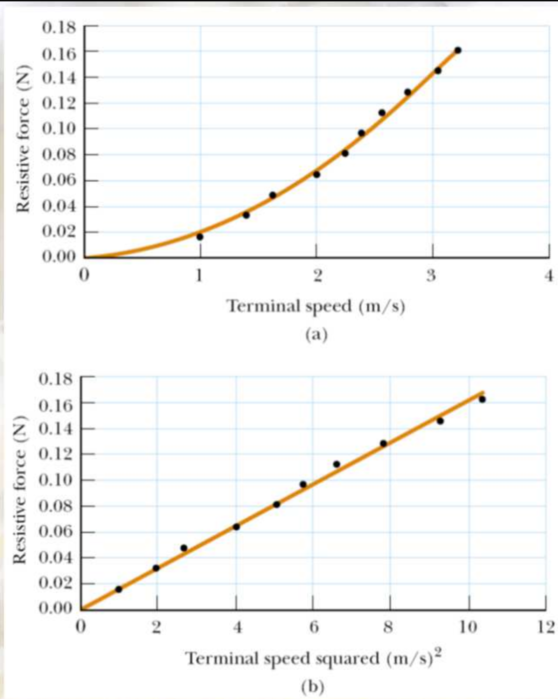
The dependence of resistive force on speed is an empirical relationship. In other words, it is based on observation rather than on a theoretical model. Imagine an experiment in which we drop a series of stacked coffee filters, and measure their terminal speeds. Table 6.2 presents data for these coffee filters as they fall through the air. The time constant τ is small, so that a dropped filter quickly reaches terminal speed. Each filter has a mass of 1.64 g. When the filters are nested together, they stack in such a way that the front-facing surface area does not increase. Determine the relationship between the resistive force exerted by the air and the speed of the falling filters.

Solution At terminal speed, the upward resistive force balances the downward gravitational force. So, a single filter falling at its terminal speed experiences a resistive force of

$$R = mg = (1.64\text{ g})\left(\frac{1\text{ kg}}{100\,0\text{ g}}\right)(9.80\text{ m/s}^2) = 0.016\,1\text{ N}$$

Two filters nested together experience 0.032 2 N of resistive force, and so forth. A graph of the resistive force on the filters as a function of terminal speed is shown in Figure 6.18a. A straight line would not be a good fit, indicating that the resistive force is *not* proportional to the speed. The behavior is more clearly seen in Figure 6.18b, in which the resistive force is plotted as a function of the square of the terminal speed. This indicates a proportionality of the resistive force to the *square* of the speed, as suggested by Equation 6.6.

Terminal Speed for Stacked Coffee Filters	
Number of Filters	v_T (m/s)[a]
1	1.01
2	1.40
3	1.63
4	2.00
5	2.25
6	2.40
7	2.57
8	2.80
9	3.05
10	3.22



Example 6.13 Resistive Force Exerted on a Baseball

A pitcher hurls a 0.145-kg baseball past a batter at 40.2 m/s (= 90 mi/h). Find the resistive force acting on the ball at this speed.

Solution We do not expect the air to exert a huge force on the ball, and so the resistive force we calculate from Equation 6.6 should not be more than a few newtons.

First, we must determine the drag coefficient D. We do this by imagining that we drop the baseball and allow it to reach terminal speed. We solve Equation 6.9 for D and substitute the appropriate values for m, v_T, and A from Table 6.1. Taking the density of air as $1.20\ \text{kg/m}^3$, we obtain

$$D = \frac{2mg}{{v_T}^2 \rho A} = \frac{2(0.145\ \text{kg})(9.80\ \text{m/s}^2)}{(43\ \text{m/s})^2(1.20\ \text{kg/m}^3)(4.2 \times 10^{-3}\ \text{m}^2)}$$
$$= 0.305$$

This number has no dimensions. We have kept an extra digit beyond the two that are significant and will drop it at the end of our calculation.

We can now use this value for D in Equation 6.6 to find the magnitude of the resistive force:

$$R = \tfrac{1}{2} D\rho A v^2$$
$$= \tfrac{1}{2}(0.305)(1.20\ \text{kg/m}^3)(4.2 \times 10^{-3}\ \text{m}^2)(40.2\ \text{m/s})^2$$
$$= \boxed{1.2\ \text{N}}$$

Example 6.14 An Object Falling in a Vacuum–Analytical Method

Consider a particle falling in a vacuum under the influence of the gravitational force, as shown in Figure 6.19. Use the analytical method to find the acceleration, velocity, and position of the particle.

Solution The only force acting on the particle is the downward gravitational force of magnitude F_g, which is also the net force. Applying Newton's second law, we set the net force acting on the particle equal to the mass of the particle times its acceleration (taking upward to be the positive y direction):

$$F_g = ma_y = -mg$$

Thus, $a_y = -g$, which means the acceleration is constant. Because $dv_y/dt = a_y$, we see that $dv_y/dt = -g$, which may be integrated to yield

$$v_y(t) = v_{yi} - gt$$

Then, because $v_y = dy/dt$, the position of the particle is obtained from another integration, which yields the well-known result

$$y(t) = y_i + v_{yi}t - \tfrac{1}{2}gt^2$$

In these expressions, y_i and v_{yi} represent the position and speed of the particle at $t_i = 0$.

Example 6.15 Euler and the Sphere in Oil Revisited

Consider the sphere falling in oil in Example 6.10. Using the Euler method, find the position and the acceleration of the sphere at the instant that the speed reaches 90.0% of terminal speed.

Solution The net force on the sphere is

$$\Sigma F = -mg + bv$$

Thus, the acceleration values in the last column of Table 6.3 are

$$a = \frac{\Sigma F(x, v, t)}{m} = \frac{-mg + bv}{m} = -g + \frac{bv}{m}$$

Choosing a time increment of 0.1 ms, the first few lines of the spreadsheet modeled after Table 6.3 look like Table 6.4. We see that the speed is increasing while the magnitude of the acceleration is decreasing due to the resistive force. We also see that the sphere does not fall very far in the first millisecond.

Further down the spreadsheet, as shown in Table 6.5, we find the instant at which the sphere reaches the speed

$0.900v_T$, which is 0.900×5.00 cm/s $= 4.50$ cm/s. This calculation shows that this occurs at $t = 11.6$ ms, which agrees within its uncertainty with the value obtained in Example 6.10. The 0.1-ms difference in the two values is due to the approximate nature of the Euler method. If a smaller time increment were used, the instant at which the speed reaches $0.900v_T$ approaches the value calculated in Example 6.10.

From Table 6.5, we see that the position and acceleration of the sphere when it reaches a speed of $0.900v_T$ are

$$y = -0.035 \text{ cm} \qquad \text{and} \qquad a = -99 \text{ cm/s}^2$$

The Sphere Begins to Fall in Oil

Step	Time (ms)	Position (cm)	Velocity (cm/s)	Acceleration (cm/s^2)
0	0.0	0.0000	0.0	−980.0
1	0.1	0.0000	−0.10	−960.8
2	0.2	0.0000	−0.19	−942.0
3	0.3	0.0000	−0.29	−923.5
4	0.4	−0.0001	−0.38	−905.4
5	0.5	−0.0001	−0.47	−887.7
6	0.6	−0.0001	−0.56	−870.3
7	0.7	−0.0002	−0.65	−853.2
8	0.8	−0.0003	−0.73	−836.5
9	0.9	−0.0003	−0.82	−820.1
10	1.0	−0.0004	−0.90	−804.0

The Sphere Reaches 0.900 v_T

Step	Time (ms)	Position (cm)	Velocity (cm/s)	Acceleration (cm/s^2)
110	11.0	−0.0324	−4.43	−111.1
111	11.1	−0.0328	−4.44	−108.9
112	11.2	−0.0333	−4.46	−106.8
113	11.3	−0.0337	−4.47	−104.7
114	11.4	−0.0342	−4.48	−102.6
115	11.5	−0.0346	−4.49	−100.6
116	11.6	−0.0351	−4.50	−98.6
117	11.7	−0.0355	−4.51	−96.7
118	11.8	−0.0360	−4.52	−94.8
119	11.9	−0.0364	−4.53	−92.9
120	12.0	−0.0369	−4.54	−91.1