The constancy of the speed of light

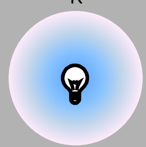
Consider the fixed system K and the moving system K′.

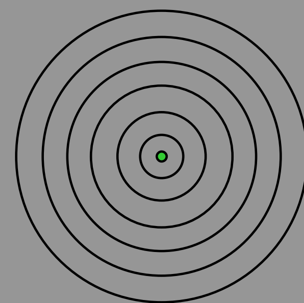
At $t = 0$, the origins and axes of both systems are coincident with system K′ moving to the right along the x axis.

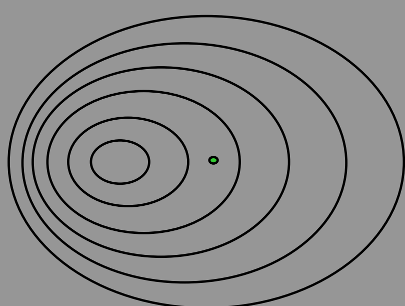
A flashbulb goes off at both origins when $t = 0$.

According to postulate 2, the speed of light will be c in both systems and the wavefronts observed in both systems must be spherical.

K







The constancy of the speed of light is not compatible with Galilean transformations.

Spherical wavefronts in K: $x^2 + y^2 + z^2 = c^2 t^2$

Spherical wavefronts in K': $x'^2 + y'^2 + z'^2 = c^2 t'^2$

Note that this cannot occur in Galilean transformations:

$$x' = x - \mathrm{v}t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

There are a couple of extra terms ($-2x\mathrm{v}t + \mathrm{v}^2t^2$) in the primed frame.

$$x'^2 + y'^2 + z'^2 = (x^2 - 2x\mathrm{v}t + \mathrm{v}^2 t^2) + y^2 + z^2 \neq c^2 t'^2$$ ☹

Finding the correct transformation

What transformation will preserve spherical wave-fronts in both frames?

Try $x' = \gamma\,(x - \mathrm{v}t)$ so that $x = \gamma'\,(x' + \mathrm{v}t')$, where γ could be anything.

By Einstein's first postulate: $\gamma' = \gamma$

The wave-front along the x'- and x-axes must satisfy: $x' = ct'$ and $x = ct$

Thus: $ct' = \gamma\,(ct - \mathrm{v}t)$ or $t' = \gamma\, t\,(1 - \mathrm{v}/c)$

and: $ct = \gamma'\,(ct' + \mathrm{v}t')$ or $t = \gamma' t' (1 + \mathrm{v}/c)$

Substituting for t in $t' = \gamma\, t\,(1 - \mathrm{v}/c)$:

$$t' = \gamma^2 t' \left(1 - \frac{\mathrm{v}}{c}\right)\left(1 + \frac{\mathrm{v}}{c}\right)$$ which yields: $$\boxed{\gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2/c^2}}}$$

γ Factoids

$$\gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2/c^2}}$$

Some simple properties of γ:

$$1 - \frac{\mathrm{v}^2}{c^2} = \frac{1}{\gamma^2}$$ which yields: $$\frac{\mathrm{v}^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

When the velocity is small:

$$\gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2/c^2}} \approx \frac{1}{1 - \frac{1}{2}\mathrm{v}^2/c^2} \approx 1 + \tfrac{1}{2}\mathrm{v}^2/c^2$$

$$\gamma \approx 1 + \tfrac{1}{2}\mathrm{v}^2/c^2$$

Finding the transformation for t'

Now substitute $x' = \gamma(x - vt)$ into $x = \gamma(x' + vt')$:

$$x = \gamma[\gamma(x - vt) + vt']$$

Solving for t' we obtain: $x - \gamma^2(x - vt) = \gamma v t'$

or: $t' = x/\gamma v - \gamma(x/v - t)$

or: $t' = \gamma t + x/\gamma v - \gamma x/v$

or: $t' = \gamma t + (\gamma x/v)(1/\gamma^2 - 1)$

$1/\gamma^2 - 1 = -v^2/c^2$

or: $$\boxed{t' = \gamma(t - vx/c^2)}$$

Lorentz Transformation Equations

$$\boxed{\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

$$\boxed{\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

$$\left(\frac{x - vt}{\sqrt{1 - v^2/c^2}}\right)^2 + y^2 + z^2 = c^2\left(\frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}\right)^2$$

$$\left(\frac{x - vt}{\sqrt{1 - v^2/c^2}}\right)^2 - c^2\left(\frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}\right)^2 + y^2 + z^2 = 0$$

$$\frac{x^2 + v^2t^2 - 2xvt}{1 - v^2/c^2} - c^2\,\frac{t^2 + v^2x^2/c^4 - 2tvx/c^2}{1 - v^2/c^2} + y^2 + z^2 = 0$$

$$\frac{x^2 + v^2t^2 - \cancel{2xvt} - c^2t^2 - v^2x^2/c^2 + \cancel{2tvx}}{1 - v^2/c^2} + y^2 + z^2 = 0$$

$$\frac{x^2 + v^2t^2 - c^2t^2 - v^2x^2/c^2}{1 - v^2/c^2} + y^2 + z^2 = 0$$

$$\frac{x^2 + v^2t^2 - c^2t^2 - v^2x^2/c^2}{1 - v^2/c^2} + y^2 + z^2 = 0$$

$$\frac{x^2\left(1 - v^2/c^2\right) - c^2t^2\left(1 - v^2/c^2\right)}{1 - v^2/c^2} + y^2 + z^2 = 0$$

$$x^2 - c^2t^2 + y^2 + z^2 = 0$$

$$x^2 + y^2 + z^2 = c^2t^2$$ ☺

Lorentz Transformation Equations

A more symmetrical form:

$$\boxed{\begin{aligned} &\beta = v/c && x' = \gamma(x - \beta ct) \\ & && y' = y \\ &\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} && z' = z \\ & && t' = \gamma(t - \beta x/c) \end{aligned}}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{cases}$$

Space and time are assumed to be homogeneous.
Thus the above transformation equations must be linear.

What's happened if the are not linear?

$$x' = a_{11}x^2$$

$$x_2' - x_1' = a_{11}\left(x_2^2 - x_1^2\right)$$

$$x_1 = 1, \quad x_2 = 2$$

$$x_2' - x_1' = a_{11}(4 - 1) = 3a_{11}$$

$$x_1 = 4, \quad x_2 = 5$$

$$x_2' - x_1' = a_{11}(25 - 16) = 9a_{11}$$ ☹

$a_{ij} \neq f(t; x, y, z)$

$a_{ij} = f(v, c)$

$a_{ij} = ?$

x axis always coincides with x' axis:

$y = 0,\ z = 0 \Rightarrow y' = 0,\ z' = 0$

Therefore:

$a_{21}, a_{24} = 0$, $a_{31}, a_{34} = 0$ $\Rightarrow$ $y' = a_{22}y + a_{23}z$, $z' = a_{32}y + a_{33}z$

xy plane always coincides with x'y' plane:

$z = 0 \Rightarrow z' = 0$

Therefore:

$a_{32} = 0$ $\Rightarrow$ $z' = a_{33}z$

xz plane always coincides with x'z' plane:

$y = 0 \Rightarrow y' = 0$

Therefore:

$a_{23} = 0$ $\Rightarrow$ $y' = a_{22}y$

A rod having L=1 coincides with y axis:

$$y_2 - y_1 = 1$$

$$y_2' - y_1' = a_{22}\left(y_2 - y_1\right) = a_{22}$$

The rod now coincides with y' axis:

$$y_2' - y_1' = 1$$

$$y_2 - y_1 = 1/a_{22}\left(y_2' - y_1'\right) = 1/a_{22}$$

Therefore:

$$1/a_{22} = a_{22} \Rightarrow a_{22} = 1$$

Similarly it can be shown that:

$$1/a_{33} = a_{33} \Rightarrow a_{33} = 1$$

Therefore for the two middle equations we have:

$$y' = y \qquad z' = z$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' = y \\ z' = z \\ t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{cases}$$

We first focus on:

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

For the 2 clocks one at (0,y,0) and another at (0,-y,0):

$$t' = a_{42}y + a_{44}t \qquad t' = -a_{42}y + a_{44}t$$

$$t' - t' = 0 = 2a_{42}y + a_{44}(t - t) = 2a_{42}y \Rightarrow a_{42} = 0$$

Similarly it can be shown that:

$$t' - t' = 0 = 2a_{43}z + a_{44}(t - t) = 2a_{43}z \Rightarrow a_{43} = 0$$

Now we focus on:

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

For all the points with x'=0, we have x=vt:

$$x' = 0 \Rightarrow x = vt \Rightarrow a_{12} = a_{13} = 0 \,\&\, a_{14} = -va_{11}$$

$$x' = a_{11}x + (-va_{11})t = a_{11}(x - vt)$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & -va_{11} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\begin{cases} x' = a_{11}(x - vt) \\ y' = y \\ z' = z \\ t' = a_{41}x + a_{44}t \end{cases}$$

$$a_{11}, a_{41}, a_{44} = ?$$

Form second postulate of relativity for a spherical pulse propagated from origin we have:

$$x^2 + y^2 + z^2 = c^2t^2 \qquad x'^2 + y'^2 + z'^2 = c^2t'^2$$

$$a_{11}^2\left(x\text{-}vt\right)^2 + y^2 + z^2 = c^2\left(a_{41}x + a_{44}t\right)^2$$

$$a_{11}^2\left(x^2 + v^2t^2 - 2xvt\right) + y^2 + z^2 =$$

$$c^2\left(a_{41}^2x^2 + a_{44}^2t^2 + 2a_{41}a_{44}xt\right)$$

$$x^2\left(a_{11}^2\text{-}a_{41}^2c^2\right) + y^2 + z^2 =$$

$$t^2\left(c^2a_{44}^2 - v^2a_{11}^2\right) + 2\left(c^2a_{41}a_{44} + va_{11}^2\right)xt$$

$$\begin{cases} a_{11}^2\text{-}a_{41}^2c^2 = 1 \\ c^2a_{44}^2 - v^2a_{11}^2 = c^2 \\ c^2a_{41}a_{44} + va_{11}^2 = 0 \end{cases}$$

$$\begin{cases} a_{11} = a_{44} = 1/\sqrt{1 - v^2/c^2} \\ a_{41} = -\dfrac{v}{c^2}/\sqrt{1 - v^2/c^2} \end{cases}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{1 - v^2/c^2} & 0 & 0 & -v/\sqrt{1 - v^2/c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{v}{c^2}/\sqrt{1 - v^2/c^2} & 0 & 0 & 1/\sqrt{1 - v^2/c^2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}} \qquad \boxed{\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{1 - v^2/c^2} & 0 & 0 & -v/\sqrt{1 - v^2/c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{v}{c^2}/\sqrt{1 - v^2/c^2} & 0 & 0 & 1/\sqrt{1 - v^2/c^2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1/\sqrt{1 - v^2/c^2} & 0 & 0 & v/\sqrt{1 - v^2/c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{v}{c^2}/\sqrt{1 - v^2/c^2} & 0 & 0 & 1/\sqrt{1 - v^2/c^2} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{1 - v^2/c^2} & 0 & 0 & -v/\sqrt{1 - v^2/c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{v}{c^2}/\sqrt{1 - v^2/c^2} & 0 & 0 & 1/\sqrt{1 - v^2/c^2} \end{pmatrix}^{-1} = \begin{pmatrix} 1/\sqrt{1 - v^2/c^2} & 0 & 0 & v/\sqrt{1 - v^2/c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{v}{c^2}/\sqrt{1 - v^2/c^2} & 0 & 0 & 1/\sqrt{1 - v^2/c^2} \end{pmatrix}$$

$$\lim_{v/c \to 0} \begin{pmatrix} 1/\sqrt{1 - v^2/c^2} & 0 & 0 & -v/\sqrt{1 - v^2/c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{v}{c^2}/\sqrt{1 - v^2/c^2} & 0 & 0 & 1/\sqrt{1 - v^2/c^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

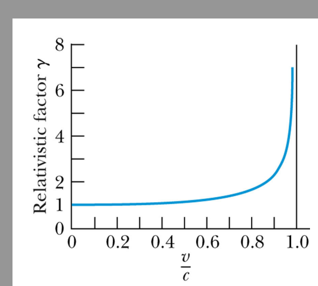
lim

v/c → 0

$$\boxed{\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}} \rightarrow \boxed{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}}$$

☺

Properties of γ



Approximate Values for γ at Various Speeds

v/c	γ
0.001 0	1.000 000 5
0.010	1.000 05
0.10	1.005
0.20	1.021
0.30	1.048
0.40	1.091
0.50	1.155
0.60	1.250
0.70	1.400
0.80	1.667
0.90	2.294
0.92	2.552
0.94	2.931
0.96	3.571
0.98	5.025
0.99	7.089
0.995	10.01
0.999	22.37

Use the Lorentz transformation equations in difference form to show that

(A) simultaneity is not an absolute concept and that

(B) a moving clock is measured to run more slowly than a clock that is at rest with respect to an observer.

Solution (A) Suppose that two events are simultaneous and separated in space such that $\Delta t' = 0$ and $\Delta x' \neq 0$ according to an observer O' who is moving with speed v relative to O. From the expression for Δt given in Equation 39.14, we see that in this case the time interval Δt measured by observer O is $\Delta t = \gamma v \, \Delta x'/c^2$. That is, the time interval for the same two events as measured by O is nonzero, and so the events do not appear to be simultaneous to O.

(B) Suppose that observer O' carries a clock that he uses to measure a time interval $\Delta t'$. He finds that two events occur at the same place in his reference frame ($\Delta x' = 0$) but at different times ($\Delta t' \neq 0$). Observer O' is moving with speed v relative to O, who measures the time interval between the events to be Δt. In this situation, the expression for Δt given in Equation 39.14 becomes $\Delta t = \gamma \Delta t'$. This is the equation for time dilation found earlier (Eq. 39.7), where $\Delta t' = \Delta t_p$ is the proper time measured by the clock carried by observer O'. Thus, O measures the moving clock to run slow.

$$\Delta x = \frac{\Delta x' + v\Delta t'}{\sqrt{1 - v^2/c^2}}$$

$$\Delta y = \Delta y'$$

$$\Delta z = \Delta z'$$

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$\Delta t = \gamma\left(\Delta t' + v\Delta x'/c^2\right)$$

$$\Delta t' = 0$$

$$\Delta t = \gamma v\Delta x'/c^2 \neq 0$$

$$\Delta x' = 0$$

$$\Delta t = \gamma\Delta t'$$