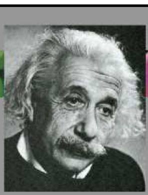
CHAPTER 2
Special Theory of Relativity 3



Albert Einstein (1879-1955)

If you are out to describe the truth, leave elegance to the tailor.

The most incomprehensible thing about the world is that it is at all comprehensible.

- Albert Einstein

▲ *Human ears have evolved to detect sound waves and interpret them as music or speech. Some animals, such as this young bat-eared fox, have ears adapted for the detection of very weak sounds. (Getty Images)*

2.10: The Doppler Effect

Christian Andreas Doppler (1803-1853)

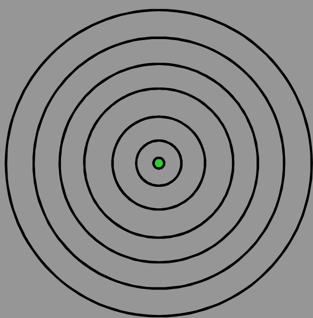
The Doppler effect for sound yields an *increased sound frequency* as a source such as a train (with whistle blowing) approaches a receiver and a *decreased frequency* as the source recedes.

A similar change in sound frequency occurs when the source is fixed and the receiver is moving.

But the formula depends on whether the source or receiver is moving.

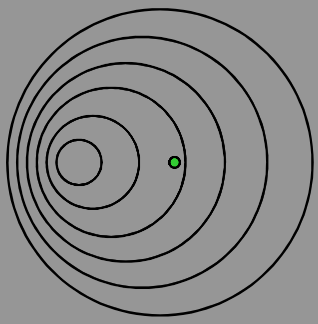
The Doppler effect in sound violates the principle of relativity because there is in fact a special frame for sound waves. Sound waves depend on media such as air, water, or a steel plate in order to propagate. Of course, light does not!

Waves from a source at rest



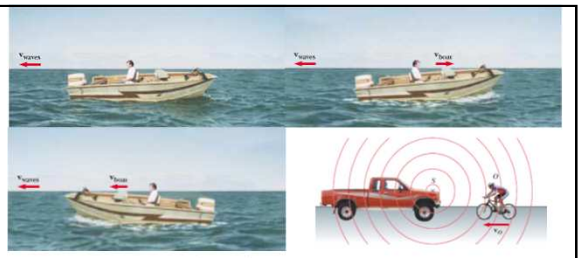
Viewers at rest everywhere see the waves with their appropriate frequency and wavelength.

Recall the Doppler Effect



A receding source yields a red-shifted wave, and an approaching source yields a blue-shifted wave.

A source passing by emits blue- then red-shifted waves.



Source moving

$$f' = \frac{v'}{\lambda}$$

$$f' = \left(\frac{v}{v - v_S}\right)f \qquad \text{(source moving toward observer)}$$

$$f' = \left(\frac{v}{v + v_S}\right)f \qquad \text{(source moving away from observer)}$$

Observer moving

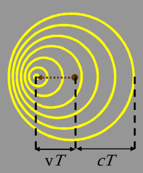
$$f' = \frac{v}{\lambda'}$$

$$f' = \left(\frac{v + v_O}{v}\right)f \qquad \text{(observer moving toward source)}$$

$$f' = \left(\frac{v - v_O}{v}\right)f \qquad \text{(observer moving away from source)}$$

The Relativistic Doppler Effect

So what happens when we throw in Relativity?



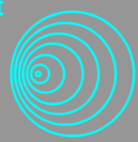
Consider a source of light (for example, a star) in system K′ receding from a receiver (an astronomer) in system K with a relative velocity v.

Suppose that (in the observer frame) the source emits N waves during the time interval T (T_0' in the source frame).

Because the speed of light is always c and the source is moving with velocity v, the total distance between the front and rear of the wave transmitted during the time interval T (in the observer frame) is:

Length of wave train = $cT + \mathrm{v}T$

The Relativistic Doppler Effect



Because there are N waves, the wavelength is given by: $\lambda = \dfrac{cT + \mathrm{v}T}{N}$

And the resulting frequency is: $\nu = \dfrac{cN}{cT + \mathrm{v}T}$

In the source frame: $N = \nu_0 T_0'$ and $T_0' = T/\gamma$ ⟵ Source frame is proper time.

Thus: $\nu = \dfrac{c[\nu_0 (T/\gamma)]}{cT + \mathrm{v}T} = \dfrac{1}{1 + \mathrm{v}/c}\dfrac{\nu_0}{\gamma} = \dfrac{\sqrt{1 - \mathrm{v}^2/c^2}}{1 + \mathrm{v}/c}\nu_0$

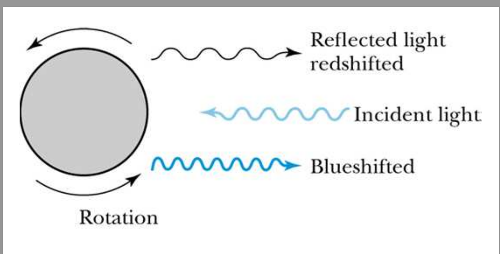
$$= \frac{\sqrt{(1 - \mathrm{v}/c)\cancel{(1 + \mathrm{v}/c)}}}{\sqrt{(1 + \mathrm{v}/c)\cancel{(1 + \mathrm{v}/c)}}}\nu_0$$

So: $\boxed{\nu = \sqrt{\dfrac{1 - \mathrm{v}/c}{1 + \mathrm{v}/c}}\,\nu_0}$

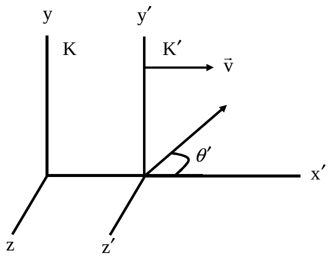
Use a + sign for v/c when the source and receiver are receding from each other and a – sign when they're approaching.

Using the Doppler shift to sense rotation

The Doppler shift has a zillion uses.



$$\exp\left(i(\vec{k}.\vec{r}-\omega t)\right) = \cos\left(\vec{k}.\vec{r}-\omega t\right) + i\sin\left(\vec{k}.\vec{r}-\omega t\right)$$



$$\cos\left(k'_x x' + k'_y y' - \omega' t'\right) = \cos\left(k'\cos\theta' x' + k'\sin\theta' y' - \omega' t'\right) =$$
$$= \cos 2\pi\left(\frac{\cos\theta' x' + \sin\theta' y'}{\lambda'} - \nu' t'\right)$$

In K' frame $\qquad \cos 2\pi\left(\frac{\cos\theta' x' + \sin\theta' y'}{\lambda'} - \nu' t'\right)$

Because Lorantz transformations are linear, in K frame the wave remains as plane wave

In K frame $\qquad \cos 2\pi\left(\frac{x\cos\theta + y\sin\theta}{\lambda} - \nu t\right)$

$$\lambda\nu = \lambda'\nu' = c$$

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \qquad\qquad t' = \frac{t - \left(v/c^2\right)x}{\sqrt{1-\beta^2}}$$

$$\cos 2\pi\left(\frac{\frac{x-vt}{\sqrt{1-\beta^2}}\cos\theta'}{\lambda'} + \frac{\sin\theta' y}{\lambda'} - \nu'\frac{t-\left(v/c^2\right)x}{\sqrt{1-\beta^2}}\right)$$

$$\cos 2\pi\left(\frac{\frac{x-vt}{\sqrt{1-\beta^2}}\cos\theta'}{\lambda'} + \frac{\sin\theta' y}{\lambda'} - \nu'\frac{t-\left(v/c^2\right)x}{\sqrt{1-\beta^2}}\right)$$

$$\cos 2\pi\left(\frac{\cos\theta'}{\lambda'\sqrt{1-\beta^2}}x - \frac{\nu'}{c}\frac{vt\cos\theta'}{\sqrt{1-\beta^2}} + \frac{\sin\theta' y}{\lambda'} - \nu'\frac{t}{\sqrt{1-\beta^2}} + \frac{c}{\lambda'}\frac{\left(v/c^2\right)x}{\sqrt{1-\beta^2}}\right)$$

$$\cos 2\pi\left(\frac{\cos\theta' + \beta}{\lambda'\sqrt{1-\beta^2}}x + \frac{\sin\theta'}{\lambda'}y - \frac{\nu'\left(\beta\cos\theta' + 1\right)}{\sqrt{1-\beta^2}}t\right)$$

$$\cos 2\pi\left(\frac{\cos\theta'+\beta}{\lambda'\sqrt{1-\beta^2}}\mathrm{x}+\frac{\sin\theta'}{\lambda'}\mathrm{y}-\frac{\nu'(\beta\cos\theta'+1)}{\sqrt{1-\beta^2}}\mathrm{t}\right)$$

$$\cos 2\pi\left(\frac{\mathrm{x}\cos\theta+\mathrm{y}\sin\theta}{\lambda}-\nu\mathrm{t}\right)$$

$$\frac{\cos\theta}{\lambda}=\frac{\cos\theta'+\beta}{\lambda'\sqrt{1-\beta^2}} \qquad \frac{\sin\theta}{\lambda}=\frac{\sin\theta'}{\lambda'}$$

$$\nu=\frac{\nu'(\beta\cos\theta'+1)}{\sqrt{1-\beta^2}}$$

$$\tan\theta=\frac{\sin\theta'\sqrt{1-\beta^2}}{\cos\theta'+\beta}$$

$$\nu=\frac{\nu'(1+\beta\cos\theta')}{\sqrt{1-\beta^2}} \qquad \nu'=\frac{\nu(1-\beta\cos\theta)}{\sqrt{1-\beta^2}}$$

$\theta'=0$ (Source and observer are going to close to each other)

$$\nu=\nu'\frac{1+\beta}{\sqrt{1-\beta^2}}=\nu'\sqrt{\frac{1+\beta}{1-\beta}}=\nu'\sqrt{\frac{\mathrm{c}+\mathrm{v}}{\mathrm{c}-\mathrm{v}}}$$

$\theta'=180$ (Source and observer are going to far from each other)

$$\nu=\nu'\frac{1-\beta}{\sqrt{1-\beta^2}}=\nu'\sqrt{\frac{1-\beta}{1+\beta}}=\nu'\sqrt{\frac{\mathrm{c}-\mathrm{v}}{\mathrm{c}+\mathrm{v}}}$$

purely relativistic transverse Doppler effect

$\theta'=90$

$$\nu=\frac{\nu'}{\sqrt{1-\beta^2}}$$

In Classical Limit

$$\nu=\frac{\nu'(1+\beta\cos\theta')}{\sqrt{1-\beta^2}}\approx\nu'(1+\beta\cos\theta')$$

$\theta'=0$ (Source and observer are going to close to each other)

$$\nu\approx\nu'(1+\beta)=\nu'(1+\mathrm{v}/\mathrm{c})=\nu'\frac{\mathrm{c}+\mathrm{v}}{\mathrm{c}}$$ ☺

$\theta'=180$ (Source and observer are going to far from each other)

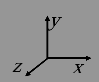
$$\nu\approx\nu'(1-\beta)=\nu'(1-\mathrm{v}/\mathrm{c})=\nu'\frac{\mathrm{c}-\mathrm{v}}{\mathrm{c}}$$ ☺

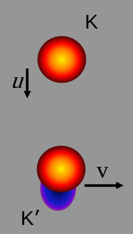
$\theta'=90$ $\qquad$ $\nu=\nu'$ ☹

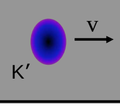
2.11: Relativistic Momentum

Because physicists believe that the conservation of momentum is fundamental, we begin by considering collisions without external forces:

$dP/dt = F_{ext} = 0$







Frank is at rest in K and throws a ball of mass m in the $-y$-direction. Mary (in the moving system) similarly throws a ball in system K′ that's moving in the x direction with velocity v with respect to system K.

Relativistic Momentum

$$u_x = \frac{\cancel{u'_x} + \mathrm{v}}{1 + \cancel{u'_x}\mathrm{v}/c^2}$$

$$u_y = \frac{u'_y}{\gamma(1 + \cancel{u'_x}\mathrm{v}/c^2)}$$

If we use the classical definition of momentum, the momentum of the ball thrown by Frank is entirely in the y direction:

$$p_{Fy} = -m\,u$$

The change of y-momentum as observed by Frank is:

$$\Delta p_{Fy} = 2\,m\,u$$

Mary measures the initial velocity of her own ball to be:

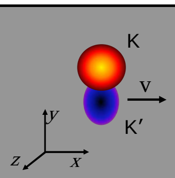
$u'_{Mx} = 0$ and $u'_{My} = u$.

In order to determine the velocity of Mary's ball, as measured by Frank, we use the relativistic velocity transformation equations:

$$u_{Mx} = \mathrm{v}$$

$$u_{My} = \frac{u}{\gamma} = u\sqrt{1 - \mathrm{v}^2/c^2}$$

Relativistic Momentum



Before the collision, the momentum of Mary's ball, as measured by Frank, becomes:

Before $p_{Mx} = m\mathrm{v}$

Before $p_{My} = mu\sqrt{1 - \mathrm{v}^2/c^2}$

For a perfectly elastic collision, the momentum after the collision is:

After $p_{Mx} = m\mathrm{v}$

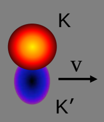
After $p_{My} = -mu\sqrt{1 - \mathrm{v}^2/c^2}$

The change in y-momentum of Mary's ball according to Frank is:

$$\Delta p_{My} = -2\,m\,u\sqrt{1 - \mathrm{v}^2/c^2}$$

whose magnitude is **different** from that of his ball: $\Delta p_{Fy} = 2\,m\,u$

Relativistic Momentum

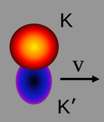


The conservation of linear momentum requires the total change in momentum of the collision, $\Delta p_F + \Delta p_M$, to be zero. The addition of these y-momenta is clearly not zero.

Linear momentum is not conserved if we use the conventions for momentum from classical physics—even if we use the velocity transformation equations from special relativity.

There is no problem with the x direction, but there is a problem with along the direction the ball is thrown in each system, the y direction.

Relativistic Momentum



Rather than abandon the conservation of linear momentum, we can make a modification of the definition of linear momentum that preserves both it and Newton's second law.

To do so requires re-examining mass to conclude that:

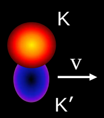
$$\boxed{\vec{p} = \gamma\, m\, \vec{\mathrm{v}}}$$ where: $$\boxed{\gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2/c^2}}}$$

Important: note that we're using γ in this formula, but the v in γ is really the velocity of the **object**, not necessarily that of its frame.

Does this modification work?

The initial y-momentum of Frank's ball is now:

$$p_{Fy} = -\gamma_F\, m u = \frac{-mu}{\sqrt{1-u_F^2/c^2}} = \frac{-mu}{\sqrt{1-u^2/c^2}}$$



The initial y-momentum of Mary's ball is now:

$$p_{My} = \gamma_M\, m u \sqrt{1-\mathrm{v}^2/c^2} = m u \frac{\sqrt{1-\mathrm{v}^2/c^2}}{\sqrt{1-u_M^2/c^2}}$$

where u_M is the speed of Mary's ball in K:

$$u_M = \sqrt{u_{Mx}^2 + u_{My}^2} = \sqrt{\mathrm{v}^2 + u^2(1-\mathrm{v}^2/c^2)}$$

from the relativistic velocity addition equations

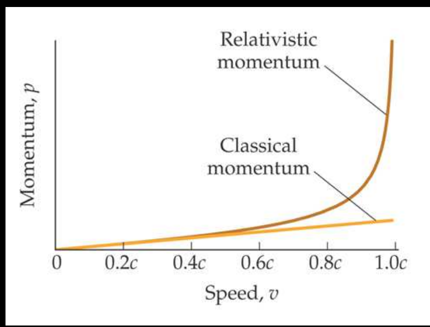
so:

after some simplification

$$p_{My} = \frac{m u\sqrt{1-\mathrm{v}^2/c^2}}{\sqrt{1-\left[\mathrm{v}^2 + u^2(1-\mathrm{v}^2/c^2)\right]/c^2}} = \frac{mu}{\sqrt{1-u^2/c^2}}$$

which perfectly cancels the y-momentum of Frank's ball

Relativistic momentum



At high velocity, does the mass increase or just the momentum?

Some physicists like to refer to the mass as the **rest mass** m_0 and call the term $m = \gamma m_0$ the **relativistic mass**. In this manner the classical form of momentum, m, is retained. The mass is then imagined to increase at high speeds.

Most physicists prefer to keep the concept of mass as an invariant, intrinsic property of an object. We adopt this latter approach and will use the term **mass** exclusively to mean **rest mass**. Although we may use the terms *mass* and *rest mass* synonymously, we will not use the term *relativistic mass*.

2.12: Relativistic Energy

We must now redefine the concepts of work and energy.

So we modify Newton's second law to include our new definition of linear momentum, and force becomes:

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{u}) = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right)}$$

where, again, we're using γ in this formula, but it's really the velocity of the **object**, not necessarily that of its frame.

Again, let's begin with classical concepts. The differential work done is:

$$dW = F\,dx = \frac{dp}{dt}dx$$

Dividing by dt:

$$\frac{dW}{dt} = \frac{dp}{dt}\frac{dx}{dt} = \frac{dp}{dt}\mathrm{v}$$

In terms of velocity derivatives:

$$\frac{dW}{d\mathrm{v}}\cancel{\frac{d\mathrm{v}}{dt}} = \frac{dp}{d\mathrm{v}}\cancel{\frac{d\mathrm{v}}{dt}}\mathrm{v}$$

Canceling the $d\mathrm{v}/dt$'s:

$$\frac{dW}{d\mathrm{v}} = \frac{dp}{d\mathrm{v}}\mathrm{v} \quad \text{or} \quad dW = \frac{dp}{d\mathrm{v}}\mathrm{v}\,d\mathrm{v}$$

Relativistic Energy

The kinetic energy will be equal to the work done starting with zero energy and ending with W_0, or from zero velocity to u.

$$K = \int_0^{W_0} dW$$

$$= \int_0^u \frac{dp}{d\mathrm{v}}\mathrm{v}\,d\mathrm{v}$$

Relativistic Energy

Integrating by parts: $K = \int_0^u \frac{dp}{d\mathrm{v}}\mathrm{v}\,d\mathrm{v} = p\mathrm{v}\Big|_0^u - \int_0^u p\,d\mathrm{v}$

$$= pu - m\int_0^u \frac{\mathrm{v}}{\sqrt{1-\mathrm{v}^2/c^2}}d\mathrm{v} = pu + mc^2\sqrt{1-\mathrm{v}^2/c^2}\Big|_0^u$$

substituting for p

because: $\frac{d}{d\mathrm{v}}\left[-c^2\sqrt{1-\mathrm{v}^2/c^2}\right] = -\cancel{\tfrac{1}{2}}c^2\frac{-\cancel{2}\mathrm{v}/\cancel{c^2}}{\sqrt{1-\mathrm{v}^2/c^2}}$

$$= \left(m\frac{u}{\sqrt{1-u^2/c^2}}\right)u + mc^2\left(\sqrt{1-u^2/c^2}-1\right)$$

$$= mc^2\left(\frac{\cancel{u^2/c^2}}{\sqrt{1-u^2/c^2}} + \frac{1-\cancel{u^2/c^2}}{\sqrt{1-u^2/c^2}} - 1\right) \qquad \boxed{K = mc^2(\gamma-1)}$$

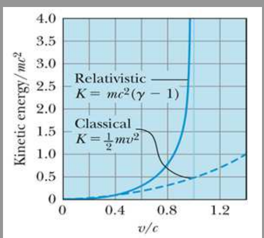
Relativistic Energy

$$\boxed{K = mc^2(\gamma-1)}$$

Written in terms of $u = \mathrm{v} << c$:

$$K \approx mc^2\left[\left(1+\tfrac{1}{2}\mathrm{v}^2/c^2\right)-1\right] = \tfrac{1}{2}m\mathrm{v}^2$$

the classical result!



Note that even an infinite amount of energy is not enough to achieve c.

Total Energy and Rest Energy

Manipulate the energy equation: $K = mc^2(\gamma-1)$

$$K = \gamma mc^2 - mc^2$$

The term mc^2 is called the **Rest Energy** and is denoted by E_0:

$$\boxed{E_0 = mc^2}$$

The sum of the kinetic and rest energies is the **total energy** of the particle E and is given by:

$$\boxed{E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \frac{E_0}{\sqrt{1-u^2/c^2}} = K + E_0}$$

Momentum and Energy

Square the momentum equation, $p = \gamma m u$, and multiply by c^2:

$$p^2c^2 = \gamma^2 m^2 u^2 c^2$$

Substituting for u^2 using $\beta^2 = u^2 / c^2$:

$$= \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) = \gamma^2 m^2 c^4 \beta^2$$

But $\beta^2 = 1 - \frac{1}{\gamma^2}$

$$p^2c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$$

And:

$$p^2c^2 = \gamma^2 m^2 c^4 - m^2 c^4$$

Momentum and Energy $p^2c^2 = \gamma^2 m^2 c^4 - m^2 c^4$

The first term on the right-hand side is just E^2, and the second is E_0^2:

$$p^2c^2 = E^2 - E_0^2$$

Rearranging, we obtain a relation between energy and momentum.

$$\boxed{E^2 = p^2c^2 + E_0^2}$$

or:

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

This equation relates the total energy of a particle with its momentum. The quantities $(E^2 - p^2c^2)$ and m are invariant quantities.

Note that when a particle's velocity is zero and it has no momentum, this equation correctly gives E_0 as the particle's total energy.

Massless particles $E^2 = p^2c^2 + m^2c^4$

We can show that any massless particle must travel at the speed of light. If $m = 0$, then:

$$E^2 = p^2c^2 \quad \text{or:} \quad E = pc$$

Noting that the total energy, E, is also given by:

$$E = \gamma mc^2$$

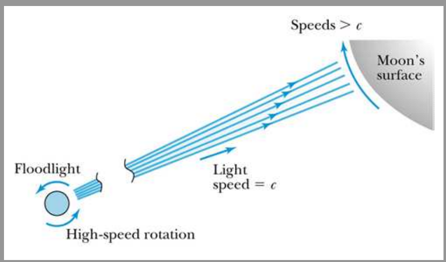
we can set the expressions for the energy equal:

$$pc = \gamma mc^2$$

But $p = \gamma m v$:

$$\gamma m v c = \gamma mc^2 \quad \text{or:} \quad \boxed{v = c}$$

Legally going faster than the speed of light



This is okay. No information is transferred.

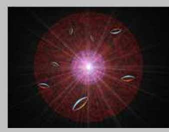
2.13: Computations in Modern Physics

We were taught in introductory physics that the international system of units is preferable when doing calculations in science and engineering.

In modern physics, a somewhat different, more convenient set of units is often used.

The smallness of quantities often used in modern physics suggests some practical changes.

The Electron Volt (eV)



Artist's rendition of an electron (don't take this too seriously)

The work done in accelerating a charge through a potential difference is given by $W = qV$. For a proton, with the charge e = 1.602 × 10^{-19} C and a potential difference of 1 V, the work done is:

$$W = (1.602 \times 10^{-19}\ \text{C})(1\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}$$

The work done to accelerate the proton across a potential difference of 1 V could also be written as:

$$W = (1\ e)(1\ \text{V}) = 1\ \text{eV}$$

Thus eV, pronounced "electron volt," is also a unit of energy. It's related to the SI (*Système International*) unit joule by:

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}$$

Rest Energy

Rest energy of a particle ($E_0 = mc^2$):

Example: E_0 (proton)

$$E_0(\text{proton}) = (1.67\times10^{-27}\ \text{kg})(3\times10^{8}\ \text{m/s})^2 = 1.50\times10^{-10}\ \text{J}$$

$$= 1.50\times10^{-10}\ \text{J}\ \frac{1\ \text{eV}}{1.602\times10^{-19}\ \text{J}} = 9.38\times10^{8}\ \text{eV}$$

Atomic mass unit (amu) (~ the number of nucleons in the nucleus):

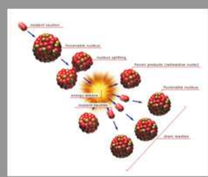
Example: carbon 12

$$\text{Mass } (^{12}\text{C atom}) = \frac{12\ \text{g/mol}}{6.02\times10^{23}\ \text{atoms/mol}}$$

$$= 1.99\times10^{-23}\ \text{g/atom}$$

$$\text{Mass } (^{12}\text{C atom}) = 1.99\times10^{-26}\ \text{kg} = 12\ \text{u/atom}$$

Binding Energy



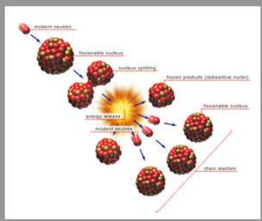
The equivalence of mass and energy becomes apparent when we study the binding energy of systems like atoms and nuclei that are formed from individual particles.

The potential energy associated with the force keeping the system together is called the **binding energy** E_B.

The binding energy is *the difference between the rest energy of the individual particles and the rest energy of the combined bound system.*

$$\boxed{M_{\text{bound system}}c^2 + E_B = \sum_i m_i c^2}$$

Fission and Fusion



Fission: Gaining energy by breaking apart a large nucleus.

$E_b < 0$ for large nuclei

Fusion: Gaining energy by fusing together small nuclei.

$E_b > 0$ for small nuclei

$E_b \sim 0$ for iron

Example:

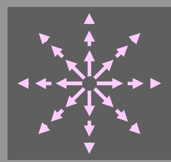
$m_{proton}\, c^2 = 938.27$ MeV

$m_{neutron}\, c^2 = 939.57$ MeV

$m_{deuteron}\, c^2 = 1875.61$ MeV

$\rightarrow\ E_B = 2.23$ MeV

Relativity and Electromagnetism



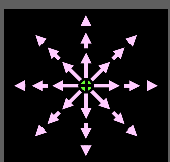
Einstein's belief that *Maxwell's equations describe electromagnetism in any inertial frame* was the key that led Einstein to the Lorentz transformations.

Maxwell's result that all electromagnetic waves travel at the speed of light led Einstein to his postulate that the speed of light is invariant in all inertial frames.

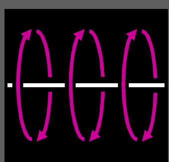
Einstein was convinced that magnetic fields appeared as electric fields when observed in another inertial frame. That conclusion is the key to electromagnetism and relativity.

But how can a magnetic field appear as an electric field simply due to motion?

Electric field lines (and hence the force field for a positive test charge) due to positive charge.



Magnetic field lines circle a current but don't affect a test charge unless it's moving.



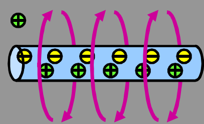
← Wire with current

How can one become the other and still give the right answer?

A Conducting Wire

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Suppose that a positive test charge and negative charges in a wire have the same velocity. And positive charges in the wire are stationary.



The electric field due to charges in the wire will be zero, so the force on the test charge will be magnetic:

$$\vec{F} = q\vec{v} \times \vec{B}$$

The magnetic field at the test charge will point into the page, so the force on the test charge will be up.

A Conducting Wire 2

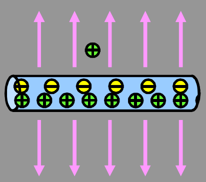
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Now transform to the frame of the previously moving charges.

Now it's the positive charges in the wire that are moving. And they will be **Lorentz-contracted**, so their density will be higher.

There will still be a magnetic field, but the test charge now has zero velocity, so its force will be zero. The excess of positive charges will yield an electric field, however:

$$\vec{F} = q\vec{E}$$



The electric field will point radially outward, and at the test charge it will point upward, so the force on the test charge will be **up**. The two cases can be shown to be identical.