Compatible Observables

Observables A and B are defined to be compatible when the corresponding operators commute:

$$[A, B] = 0$$

Observables A and B are defined to be incompatible when the corresponding operators do not commute:

$$[A, B] \neq 0$$

For example, S^2 and S_z are compatible observables, while S_x and S_z are incompatible observables.

$$[S^2, S_i] = 0$$

$$[S_x, S_z] = -i\hbar S_y$$

Theorem. Suppose that A and B are compatible observables, and the eigenvalues of A are nondegenerate. Then the matrix elements <a``|B|a`> are all diagonal. (Recall here that the matrix elements of A are already diagonal if {|a`>} are used as the base kets.)

$$\langle a''|[A, B]|a'\rangle = (a'' - a')\langle a''|B|a'\rangle = 0$$

So <a``|B|a`> must vanish unless a'' = a' which proves our assertion.

$$\langle a''|[A,B]|a'\rangle = (a''-a')\langle a''|B|a'\rangle = 0$$

$$\langle a''|B|a'\rangle = \delta_{a'a''}\langle a'|B|a'\rangle$$

So both A and B can be represented by diagonal matrices with the same set of base kets.

We can write:

$$B = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|$$

Suppose that B acts on an eigenstate of A

$$B|a'\rangle = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|a'\rangle = (\langle a'|B|a'\rangle)|a'\rangle$$

$$b' \equiv \langle a'|B|a'\rangle$$

But this is nothing other than the eigenvalue equation for the operator B with eigenvalue b`

The ket |a'> is therefore a simultaneous eigenket of A and B.

Just to be impartial to both operators, we may use |a`,b`> to characterize this simultaneous eigenket.

We have seen that compatible observables have simultaneous eigenkets.

Even though the proof given is for the case where the A eigenkets are nondegenerate, the statement holds even if there is an n-fold degeneracy, that is,

$$A|a'^{(i)}\rangle = a'|a'^{(i)}\rangle \quad \text{for } i = 1, 2, \ldots, n$$

A simultaneous eigenket of A and B, denoted by |a`,b`>, has the property

$$A|a', b'\rangle = a'|a', b'\rangle$$

$$B|a', b'\rangle = b'|a', b'\rangle$$

We can obviously generalize our considerations to a situation where there are several (more than two) mutually compatible observables, namely,

$$[A, B] = [B, C] = [A, C] = \cdots = 0$$

قضیه: شرط لازم و کافی برای آن که دو اپراتور هرمیتی دارای ویژه توابع مشترک باشند آن است که با یکدیگر جابه جا شوند.

برهان شرط لازم:

$$Bu_a(x) = bu_a(x) \qquad Au_a(x) = au_a(x)$$

$$ABu_a(x) = Abu_a(x) = bAu_a(x) = bau_a(x)$$

$$= abu_a(x) = aBu_a(x) = Bau_a(x) = BAu_a(x)$$

$$(AB - BA)u_a(x) = 0 \qquad ABu_a(x) = BAu_a(x)$$

البته اگر رابطه فوق فقط برای یک $u_a(x)$ برقرار باشد نمی تواند خیلی جالب باشد، اما اگر برای یک مجموعه پایه کامل $u_a(x)$ صادق باشد یعنی برانی همه توابع مجذورا انتگرال

$$\Psi(x) = \sum_a C_a u_a(x)$$

آنگاه:

$$\sum_a (AB - BA) u_a(x) = (AB - BA) \sum_a C_a u_a(x) = 0$$

$$[A, B] = 0$$

برهان شرط کافی:

به عکس اگر دو عملگر **A** و **B** با هم جا به جا شوند، یعنی:

$$A(Bu_a(x)) = a(Bu_a(x))$$

$$ABu_a(x) = BAu_a(x) = aBu_a(x)$$

یعنی **$Bu_a(x)$** هم یک ویژه تابع عملگر **A** است (علاوه بر خود **$u_a(x)$**) با همان ویژه مقدار **a**

حال دو حالت تبهگن و غیر تبهگن را جداگانه بحث می کنیم تا ببینیم که از رابطه زیر:

$$A(Bu_a(x)) = a(Bu_a(x))$$

آیا می توان نتیجه گرفت که **A** و **B** ویژه توابع مشترک دارند.

حالت 1) حالت غیر تبهگن: اگر **A** تبهگن نباشد آنگاه فقط یک ویژه تابع **$u_a(x)$** متناظر با ویژه مقدار **a** خواهد بود، پس:

$$A(Bu_a(x)) = a(Bu_a(x))$$

$$Au_a(x) = au_a(x)$$

$$\Rightarrow \quad Bu_a(x) = bu_a(x)$$

و این بدان معنی است که **A** و **B** ویژه توابع مشترک **$u_a(x)$** را دارند.

و این یعنی این که ویژه توابع **B** به دست آمد.

حالت 2) حالت تبهگن:

$$m+1 \ldots\ldots\ldots\ldots\ldots m+n$$

$$\ldots\ldots\ldots\ldots\ldots m$$

تعداد حالتهای تبهگن

تعداد حالتهای غیر تبهگن

در بعد غیر تبهگن مشکلی نیست اما در زیر فضای تبهگن چگونه می توان ویژه بردارهای عملگر **B** را به طور همزمان ویژه بردارهای عملگر **A** نیز بتوانند باشند یافت؟

در این حالت نمی توان به راحتی رابطه زیر را به کار برد:

$$Bu_a(x) = bu_a(x)$$

$Bu_a(x)$ باید ویژه تابع **A** باشد زیرا $A(Bu_a(x))=a(Bu_a(x))$ ولی به دلیل تبهگنی یک ترکیب خطی از آنها را می گیریم:

$$Bu^{(i)}{}_a(x) = \sum_{k=m+1}^{m+n} b_{ki} u^{(k)}{}_a(x)$$

در زیر فضای تبهگن (m+1 to m+n) بردار ویژه B را به صورت زیر داریم:

$$v^{(j)}{}_a(x) = \sum_{l=m+1}^{m+n} d_{jl} u^{(l)}{}_a(x)$$

$$\sum_{l=m+1}^{m+n} (d_{jl} b_{kl} - \lambda_j d_{jl} \delta_{kl}) = 0$$

$$B v^{(j)}{}_a(x) = \lambda_j v^{(j)}{}_a(x)$$

$$\sum_{l=m+1}^{m+n} (b_{kl} - \lambda_j \delta_{kl}) d_{jl} = 0$$

$$B \sum_{l=m+1}^{m+n} d_{jl} u^{(l)}{}_a(x) = \lambda_j \sum_{l=m+1}^{m+n} d_{jl} u^{(l)}{}_a(x)$$

$$\sum_{\substack{l=m+1 \\ k=m+1}}^{m+n} d_{jl} b_{kl} u^{(k)}{}_a(x) = \lambda_j \sum_{l=m+1}^{m+n} d_{jl} u^{(l)}{}_a(x) = \lambda_j \sum_{k=m+1}^{m+n} d_{jk} u^{(k)}{}_a(x)$$

$$\sum_{l=m+1}^{m+n} d_{jl} b_{kl} u^{(k)}{}_a(x) = \lambda_j d_{jk} u^{(k)}{}_a(x)$$

$$\sum_{l=m+1}^{m+n} d_{jl} b_{kl} = \lambda_j d_{jk}$$

شرط وجود جواب غیر بدیهی:

$$\det(b_{kl} - \lambda_j \delta_{kl}) = 0$$

که از حل آن λ_j ها و سپس d_{jl} به دست می آیند. در نتیجه $v_a^{(j)}(x)$ ها معلوم می شوند. بنابراین بردارهای ویژه **B** معلوم شده اند.

حتی پس از یافتن ویژه توابع A و سپس ترکیب آنها و به دست آوردن ویژه توابع عملگر جابه جا شونده با A، یعنی B، هنوز ممکن است تبهگنی وجود داشته باشد یعنی چندین ویژه تابع A و B به طور همزمان وجود داشته باشند که ویژه توابع عملگر دیگری مانند C باشند که با عملگرهای A و B جابه جا شوند و توابع می توانند باز ترکیب شوند تا ویژه توابع مشترک A، B و C باشند که ویژه مقادیر آنها تبهگنی A و B را روشن سازد. این کار را می توان تا آنجا ادامه

مجموعه عملگرهایی که باهم جابه جا می شوند و ویژه توابع مشترک دارند یک مجموعه کامل تشکیل می دهند که به آنها مشاهده پذیرهای جابه جا شونده گویند.

$$A u_{ab\ldots m}(x) = a u_{ab\ldots m}(x)$$

$$[A, B] = [A, C] = \ldots = [A, M] = 0$$

$$B u_{ab\ldots m}(x) = b u_{ab\ldots m}(x)$$

$$[B, C] = [B, D] = \ldots = [B, M] = 0$$

....

$$M u_{ab\ldots m}(x) = m u_{ab\ldots m}(x)$$

....

....

We now consider measurements of A and B when they are compatible observables.

Suppose we measure A first and obtain result a`.

Subsequently, we may measure B and get result b'.

Finally we measure A again.

It follows from our measurement formalism that the third measurement always gives a' with certainty, that is, the second (B) measurement does not destroy the previous information obtained in the first (A) measurement.

This is rather obvious when the eigenvalues of A are nondegenerate:

$$|\alpha\rangle \xrightarrow{A \text{ measurement}} |a', b'\rangle \xrightarrow{B \text{ measurement}} |a', b'\rangle \xrightarrow{A \text{ measurement}} |a', b'\rangle$$

When there is degeneracy, the argument goes as follows:

After the first (A) measurement, which yields a` the system is thrown into some linear combination

$$\sum_{i}^{n} c_{a'}^{(i)} |a', b^{(i)}\rangle$$

where n is the degree of degeneracy and the kets $|a`,b^{(i)}>$ all have the same eigenvalue a' as far as operator A is concerned.

The second (B) measurement may select just one of the terms in the linear combination , say, $|a`,b^{(j)}>$, but the third (A) measurement applied to it still yields a'.

The term compatible is indeed deemed appropriate.

Incompatible Observables

We now turn to incompatible observables, which are more nontrivial.

The first point to be emphasized is that incompatible observables do not have a complete set of simultaneous eigenkets.

To show this let us assume the converse to be true.

There would then exist a set of simultaneous eigenkets with property:

$$A|a',b'\rangle = a'|a',b'\rangle \qquad B|a',b'\rangle = b'|a',b'\rangle$$

Clearly, we have:

$$AB|a', b'\rangle = Ab'|a', b'\rangle = a'b'|a', b'\rangle$$

$$BA|a', b'\rangle = Ba'|a', b'\rangle = a'b'|a', b'\rangle$$

Hence

$$AB|a', b'\rangle = BA|a', b'\rangle$$

and thus [A,B] = 0 in contradiction to the assumption.

So in general, |a`, b`> does not make sense for incompatible observables.

There is, however, an interesting exception:

It may happen that there exists a subspace of the ket space such that

$$AB|a', b'\rangle = BA|a', b'\rangle$$

holds for all elements of this subspace, even though A and B are incompatible.

An example from the theory of orbital angular momentum may be helpful here.

Suppose we consider an $l = 0$ state (s-state).

Even though Lx and Lz do not commute, this state is a simultaneous eigenstate of Lx and Lz (with eigenvalue zero for both operators).

$$L_z Y_{lm} = m\hbar Y_{lm}$$

$$L_\pm = L_x \pm iL_y$$

$$L_x = \frac{L_+ + L_-}{2}$$

$$L_\pm Y_{lm} = \hbar\sqrt{l(l+1) - m(m\pm 1)}\, Y_{l,m\pm 1}$$

$$L_z Y_{00} = (0)\hbar Y_{00} = 0$$

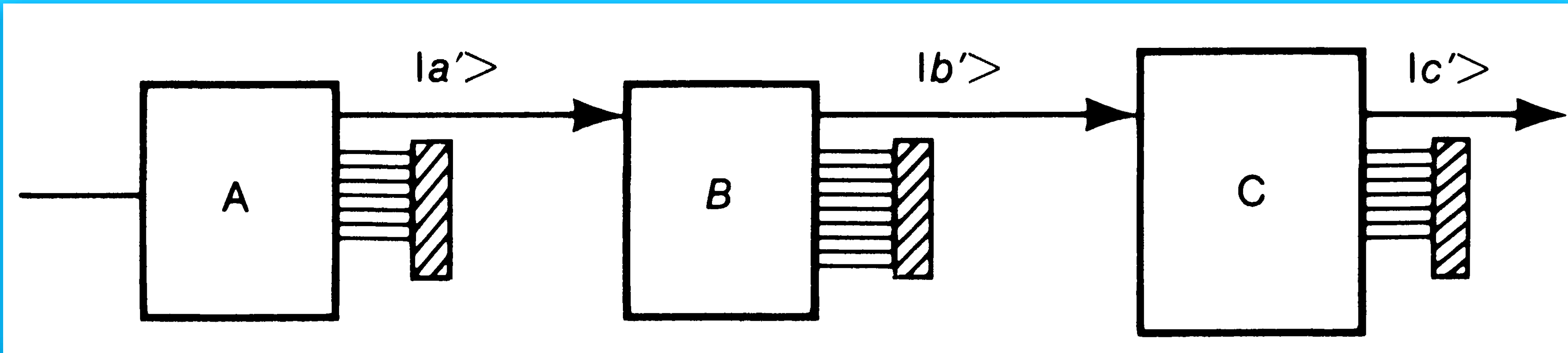
$$L_x Y_{00} = 0$$

The subspace in this case is one-dimensional.

We already encountered some of the peculiarities associated with incompatible observables when we discussed sequential Stern-Gerlach experiments.

We now give a more general discussion of experiments of that type.

Consider the sequence of selective measurements shown in following Figure.



The first (A) filter selects some particular |a'> and rejects all others, the second (B) filter selects some particular |b'> and rejects all others, and the third (C) filter selects some particular |c'> and rejects all others.

We are interested in the probability of obtaining |c'> when the beam coming out of the first filter is normalized to unity.

Because the probabilities are multiplicative, we obviously have

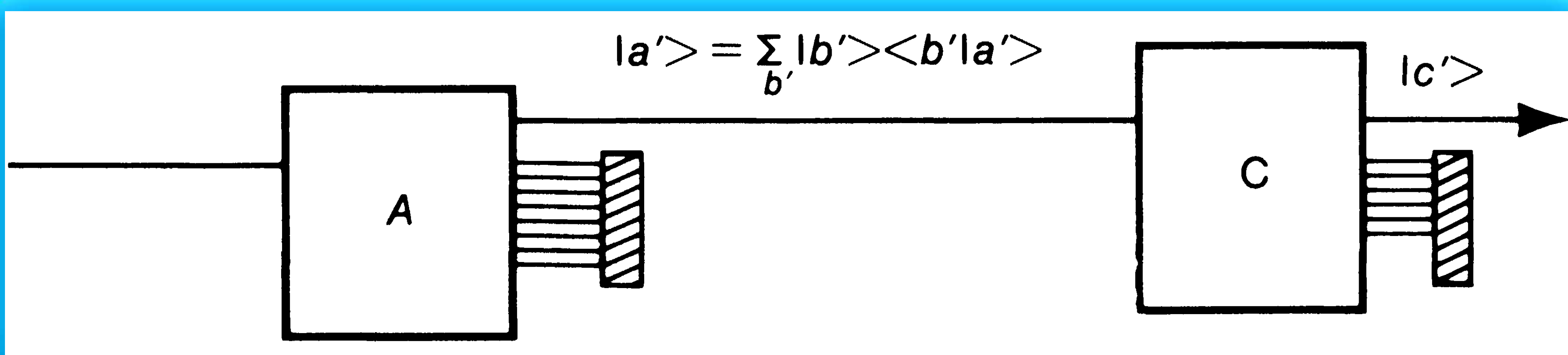
$$|\langle c'|b'\rangle|^2|\langle b'|a'\rangle|^2$$

Now let us sum over b' to consider the total probability for going through all possible b' routes.

$$\sum_{b'}|\langle c'|b'\rangle|^2|\langle b'|a'\rangle|^2=\sum_{b'}\langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle$$

Operationally this means that we first record the probability of obtaining c' with all but the first b' route blocked, then we repeat the procedure with all but the second b' blocked, and so on; then we sum the probabilities at the end.

We now compare this with a different arrangement, where the B filter is absent (or not operative); see following Figure.



Clearly, the probability is just $|\langle c'|a'\rangle|^2$, which can also be written as follows:

$$|\langle c'|a'\rangle|^2 = \Big|\sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\Big|^2 = \sum_{b'}\sum_{b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle$$

Notice that above expression is different from the following expression!

$$\sum_{b'} |\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2 = \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle$$

This is remarkable because in both cases the pure $|a'\rangle$ beam coming out of the first (A) filter can be regarded as being made up of the B eigenkets

$$|a'\rangle = \sum_{b'} |b'\rangle\langle b'|a'\rangle$$

where the sum is over all possible values of b.

The crucial point to be noted is that the result coming out of the C filter depends on whether or not B measurements have actually been carried out.

In the first case we experimentally ascertain which of the B eigenvalues are actually realized; in the second case, we merely imagine |a'> to be built up of the various |b'>'s in the sense of the above equation.

Put in another way, actually recording the probabilities of going through the various b' routes makes all the difference even though we sum over b' afterwards.

Here lies the heart of quantum mechanics.

Under what conditions do the two expressions become equal?

It is left as an exercise for the reader to show that for this to happen, in the absence of degeneracy, it is sufficient that

$$[A, B] = 0 \quad \text{or} \quad [B, C] = 0$$


In other words, the peculiarity we have illustrated is characteristic of incompatible observables.

اصل عدم قطعیت هایزنبرگ

برای اثبات رابطه فوق ابتدا سه لم زیر را اثبات می کنیم:

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|<[A,B]>|^2$$

لم 1: نامساوی شوارتز

$$<\alpha|\alpha><\beta|\beta> \geq |<\alpha|\beta>|^2$$

$$\lambda = -\frac{<\beta|\alpha>}{<\beta|\beta>}$$

$$(<\alpha| + \lambda^* <\beta|).(|\alpha> + \lambda|\beta>) \geq 0$$

$$<\alpha|\alpha> + \lambda<\alpha|\beta> + \lambda^*<\beta|\alpha> + \lambda^2<\beta|\beta> \geq 0$$

$$<\alpha|\alpha> - \frac{<\beta|\alpha><\alpha|\beta>}{<\beta|\beta>} - \frac{<\alpha|\beta><\beta|\alpha>}{<\beta|\beta>} + \frac{<\beta|\alpha><\beta|\alpha><\beta|\beta>}{<\beta|\beta>} \geq 0$$

$$<\alpha|\alpha><\beta|\beta> - <\beta|\beta> - <\alpha|\alpha> + <\beta|\alpha>^2 \geq 0$$

لم 2: مقدار چمشداشتی یک عملگر هرمیتی حقیقی خالص است

$$<\alpha|A|\beta> = <\beta|A^\dagger|\alpha>^* = <\beta|A|\alpha>^*$$

$$|\beta> \to |\alpha>$$

$$<\alpha|A|\alpha> = <A> = <\alpha|A|\alpha>^* = <A>^*$$

لم 3: مقدار چمشداشتی یک عملگر پاد هرمیتی موهومی خالص است

$$<\alpha|A|\alpha> = <\alpha|A^\dagger|\alpha>^* = -<\alpha|A|\alpha>^* = -<A>^*$$

حال به اثبات اصل عدم قطعیت می پردازیم:

$$|\alpha> = \Delta A|>$$

$$|\beta> = \Delta B|>$$

کتهای کاملا دلخواه

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|<[A,B]>|^2$$

$$\Delta A \Delta B = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}$$

$$[A,B]^\dagger = (AB - BA)^\dagger = BA - AB = -[A,B]$$

$$[\Delta A, \Delta B] = \Delta A \Delta B - \Delta B \Delta A = (A - <A>)(B - <B>) - (B - <B>)(A - <A>)$$

$$= (AB - \cancel{A<B>} - \cancel{<A>B} + \cancel{<A><B>}) - (BA - \cancel{B<A>} - \cancel{<B>A} + \cancel{<B><A>})$$

$$[\Delta A, \Delta B] = AB - BA = [A,B]$$

$$<\Delta A \Delta B> = \frac{1}{2} < \boxed{[A,B]} > + \frac{1}{2} < \boxed{\{A,B\}} >$$

موهومی خالص

حقیقی خالص

$$<(\Delta A)^2><(\Delta B)^2 \geq \frac{1}{4} |<[A,B]>|^2 + \frac{1}{4} |<\{A,B\}>|^2$$

$$<(\Delta A)^2><(\Delta B)^2 \geq \frac{1}{4} |<\boxed{[A,B]}>|^2$$

موهومی خالص

پس اگر نخواهیم علامت قدرمطلق را حفظ کنیم باید یک **i** اضافه نماییم.

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \langle i[A,B] \rangle^2$$