Momentum as a Generator of Translation

What is the physical significance we can attach to K?

J. Schwinger, lecturing on quantum mechanics, once remarked, "... for fundamental properties we will borrow only names from classical physics."

An infinitesimal translation in classical mechanics can be regarded as a canonical transformation,

$$\mathbf{x}_{\text{new}} \equiv \mathbf{X} = \mathbf{x} + d\mathbf{x} \qquad \mathbf{p}_{\text{new}} \equiv \mathbf{P} = \mathbf{p}$$

obtainable from the generating function (Goldstein 1980, 395 and 411).

$$F(\mathbf{x}, \mathbf{P}) = \mathbf{x} \cdot \mathbf{P} + \mathbf{p} \cdot d\mathbf{x}$$

Can the K operator be identified with the momentum operator itself?

Unfortunately the dimension is all wrong; the K operator has the dimension of 1/length because **K.dx'** must be dimensionless.

$$\mathbf{K} = \frac{\mathbf{p}}{\text{universal constant with the dimension of action}}$$

$$\frac{2\pi}{\lambda} = \frac{p}{\hbar}$$

$$\mathbf{K} = \frac{\mathbf{p}}{\hbar}$$

$$\mathscr{T}(d\mathbf{x}') = 1 - i\mathbf{K}\cdot d\mathbf{x}'$$

$$\mathscr{T}(d\mathbf{x}') = 1 - i\mathbf{p}\cdot d\mathbf{x}'/\hbar$$

$$[\mathrm{x}_i, \mathrm{K}_j] = i\delta_{ij}$$

$$[x_i, p_j] = i\hbar\delta_{ij}$$

So far we have concerned ourselves with infinitesimal translations.

Let us consider a finite translation in the x-direction by an amount Δx':

$$\mathscr{T}(\Delta x'\hat{\mathbf{x}})|\mathbf{x}'\rangle = |\mathbf{x}' + \Delta x'\hat{\mathbf{x}}\rangle$$

$$\Delta x' = \lim_{N\to\infty} N\frac{\Delta x'}{N}$$

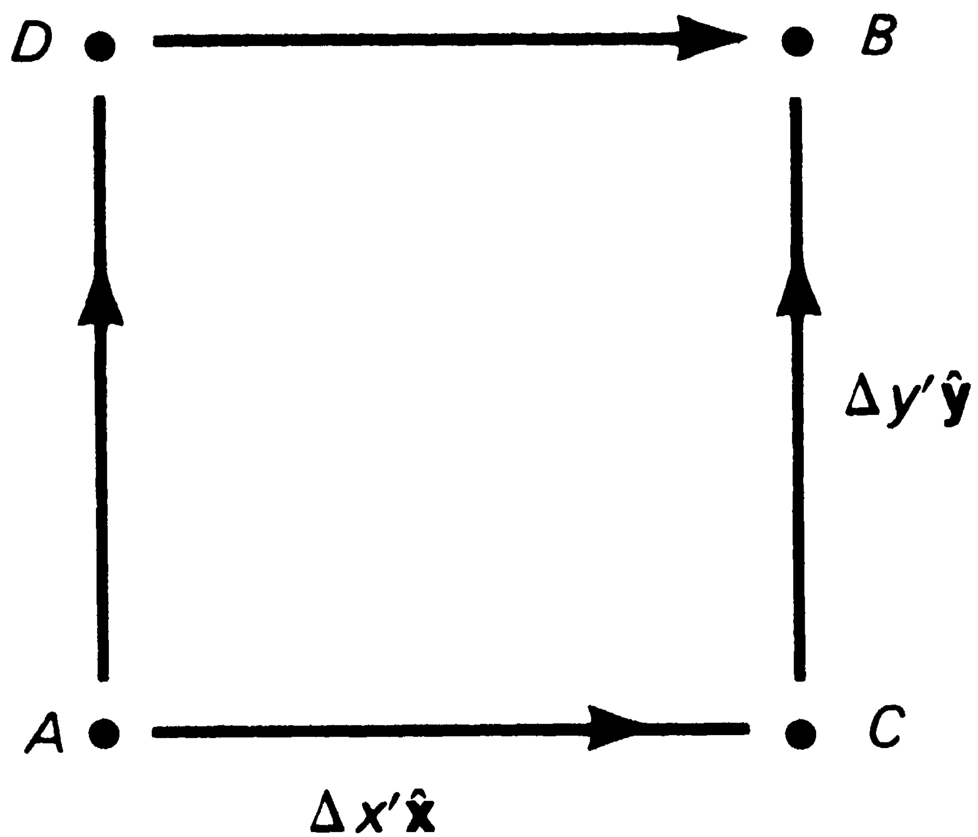
$$\mathrm{T}(\Delta x'\hat{x}) = \lim_{N\to\infty} \overbrace{\mathrm{T}\left(\frac{\Delta x'}{N}\hat{x}\right)\mathrm{T}\left(\frac{\Delta x'}{N}\hat{x}\right)\ldots\mathrm{T}\left(\frac{\Delta x'}{N}\hat{x}\right)}^{N\ times}$$

$$\mathscr{T}(\Delta x'\hat{\mathbf{x}}) = \lim_{N\to\infty}\left(1 - \frac{ip_x\Delta x'}{N\hbar}\right)^N$$

$$= \exp\left(-\frac{ip_x\Delta x'}{\hbar}\right)$$

$$\exp(X) \equiv 1 + X + \frac{X^2}{2!} + \cdots$$

A fundamental property of translations is that successive translations in different directions, say in the x- and y-directions, commute.



Graphically, in shifting from A and B it does not matter whether we go via C or via D.

$$\mathscr{T}(\Delta y'\hat{\mathbf{y}})\mathscr{T}(\Delta x'\hat{\mathbf{x}}) = \mathscr{T}(\Delta x'\hat{\mathbf{x}} + \Delta y'\hat{\mathbf{y}})$$

$$\mathscr{T}(\Delta x'\hat{\mathbf{x}})\mathscr{T}(\Delta y'\hat{\mathbf{y}}) = \mathscr{T}(\Delta x'\hat{\mathbf{x}} + \Delta y'\hat{\mathbf{y}})$$

Mathematically.

This point is not so trivial as it may appear; we will show in Chapter 3 that rotations about different axes do not commute.

Treating up to second order, we obtain

$$[\mathscr{T}(\Delta y'\hat{\mathbf{y}}), \mathscr{T}(\Delta x'\hat{\mathbf{x}})] = \left[\left(1 - \frac{ip_y\Delta y'}{\hbar} - \frac{p_y^2(\Delta y')^2}{2\hbar^2} + \cdots\right),\right.$$

$$\boxed{[p_x, p_y] = 0}$$

$$\left.\left(1 - \frac{ip_x\Delta x'}{\hbar} - \frac{p_x^2(\Delta x')^2}{2\hbar^2} + \cdots\right)\right]$$

$$\boxed{[p_i, p_j] = 0}$$

$$[\mathscr{T}(\Delta y'\hat{\mathbf{y}}), \mathscr{T}(\Delta x'\hat{\mathbf{x}})] = 0$$

$$\simeq -\frac{(\Delta x')(\Delta y')[p_y, p_x]}{\hbar^2}$$

$$[p_i, p_j] = 0$$

px, py, and pz are mutually compatible observables.

We can therefore conceive of a simultaneous eigenket of Px,Py,Pz namely,

$$|\mathbf{p}'\rangle \equiv |p'_x, p'_y, p'_z\rangle$$

$$p_x|\mathbf{p}'\rangle = p'_x|\mathbf{p}'\rangle \qquad p_y|\mathbf{p}'\rangle = p'_y|\mathbf{p}'\rangle \qquad p_z|\mathbf{p}'\rangle = p'_z|\mathbf{p}'\rangle$$

$$\mathscr{T}(d\mathbf{x}')|\mathbf{p}'\rangle = \left(1 - \frac{i\mathbf{p}\cdot d\mathbf{x}'}{\hbar}\right)|\mathbf{p}'\rangle = \left(1 - \frac{i\mathbf{p}'\cdot d\mathbf{x}'}{\hbar}\right)|\mathbf{p}'\rangle$$

We see that the momentum eigenket remains the same even though it suffers a slight phase change, so unlike |x'>, |p'> is an eigenket of T(dx'), which we anticipated because

$$[\mathbf{p}, \mathscr{T}(d\mathbf{x}')] = 0$$

Notice, however, that the eigenvalue of T(dx') is complex; we do not expect a real eigenvalue here because T(dx') though unitary, is not Hermitian.

The Canonical Commutation Relations

We summarize the commutator relations we inferred by studying the properties of translation:

$$[x_i, x_j] = 0 \qquad [p_i, p_j] = 0 \qquad [x_i, p_j] = i\hbar\delta_{ij}$$

They are called "fundamental quantum conditions" or "**canonical commutation relations**", or the "fundamental commutation relations".

Also in 1925, P. A. M. Dirac observed that the various quantum mechanical relations can be obtained from the corresponding classical relations just by replacing classical Poisson brackets by commutators, as follows:

$$[\ ,\ ]_{\text{classical}} \to \frac{[\ ,\ ]}{i\hbar}$$

$$[A(q,p), B(q,p)]_{\text{classical}} \equiv \sum_s \left(\frac{\partial A}{\partial q_s}\frac{\partial B}{\partial p_s} - \frac{\partial A}{\partial p_s}\frac{\partial B}{\partial q_s} \right)$$

For example, in classical mechanics, we have

$$[x_i, p_j]_{\text{classical}} = \delta_{ij}$$

$$[x_i, p_j] = i\hbar\delta_{ij}$$

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2$$

$$\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \hbar^2/4$$

In particular, the following relations can be proved regardless of whether [,] is understood as a classical Poisson bracket or as a quantum-mechanical commutator:

$$[A, A] = 0$$

$$[A, B] = -[B, A]$$

$$[A, c] = 0 \qquad (c \text{ is just a number})$$

$$[A + B, C] = [A, C] + [B, C]$$

$$[A,[B, C]] + [B,[C, A]] + [C,[A, B]] = 0$$

The last relation is known as the Jacobi identity.

WAVE FUNCTIONS IN POSITION AND MOMENTUM SPACE

Position-Space Wave Function

$$x|x'\rangle = x'|x'\rangle$$

$$\langle x''|x'\rangle = \delta(x''-x')$$

$$|\alpha\rangle = \int dx'|x'\rangle\langle x'|\alpha\rangle$$

$$|\langle x'|\alpha\rangle|^2 dx'$$

the probability for the particle to be found in a narrow interval dx‘ around x’.

$$\langle x'|\alpha\rangle = \psi_\alpha(x')$$

$$\langle \beta|\alpha\rangle = \int dx' \langle \beta|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \psi_\beta^*(x')\psi_\alpha(x')$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$u_{a'}(x') = \langle x'|a'\rangle$$

$$\langle x'|\alpha\rangle = \sum_{a'} \langle x'|a'\rangle\langle a'|\alpha\rangle$$

$$\psi_\alpha(x') = \sum_{a'} c_{a'} u_{a'}(x')$$

$$\langle\beta|A|\alpha\rangle = \int dx' \int dx'' \langle\beta|x'\rangle\langle x'|A|x''\rangle\langle x''|\alpha\rangle$$

$$= \int dx' \int dx'' \psi_\beta^*(x')\langle x'|A|x''\rangle\psi_\alpha(x'')$$

$$A = x^2$$

$$\langle x'|x^2|x''\rangle = (\langle x'|)\cdot(x''^2|x''\rangle) = x'^2\delta(x'-x'')$$

$$\langle\beta|x^2|\alpha\rangle = \int dx' \langle\beta|x'\rangle x'^2\langle x'|\alpha\rangle$$

$$= \int dx' \psi_\beta^*(x') x'^2 \psi_\alpha(x')$$

Note that the f(x) on the left-hand side is an operator, while the f(x') on the right-hand side is not an operator.

Or more generally

$$\langle\beta|f(x)|\alpha\rangle = \int dx' \psi_\beta^*(x') f(x') \psi_\alpha(x')$$

Momentum Operator in the Position Basis

We now examine how the momentum operator may look in the x-basis.

Our starting point is the definition of momentum as the generator of infinitesimal translations:

$$\left(1-\frac{ip\Delta x'}{\hbar}\right)|\alpha\rangle = \int dx' \mathscr{T}(\Delta x')|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' |x'+\Delta x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' |x'\rangle\langle x'-\Delta x'|\alpha\rangle$$

$$= \int dx' |x'\rangle\left(\langle x'|\alpha\rangle - \Delta x' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$p|\alpha\rangle = \int dx' |x'\rangle\left(-i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

Thus for the matrix element p in the x-representation, we obtain

$$\langle x'|p|x''\rangle = -i\hbar\frac{\partial}{\partial x'}\delta(x'-x'')$$

$$p|\alpha\rangle = \int dx' |x'\rangle \left(-i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \right)$$

$$\langle\beta|p|\alpha\rangle = \int dx' \langle\beta|x'\rangle \left(-i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \right)$$

$$= \int dx' \psi_\beta^*(x') \left(-i\hbar \frac{\partial}{\partial x'} \right) \psi_\alpha(x')$$

$$\langle\beta|p^n|\alpha\rangle = \int dx' \psi_\beta^*(x') (-i\hbar)^n \frac{\partial^n}{\partial x'^n} \psi_\alpha(x')$$

Momentum Space

$$p|p'\rangle = p'|p'\rangle$$

$$\langle p'|p''\rangle = \delta(p'-p'')$$

$$|\langle p'|\alpha\rangle|^2 dp'$$

The probability that a measurement of p gives eigenvalue p' within a narrow interval dp‘.

$$\langle p'|\alpha\rangle = \phi_\alpha(p')$$

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)$$

$$\int dp'\langle\alpha|p'\rangle\langle p'|\alpha\rangle = \int dp'|\phi_\alpha(p')|^2 = 1$$

$$\langle x'|p|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

$$p'\langle x'|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

$$\langle x'|p'\rangle = N\exp\left(\frac{ip'x'}{\hbar}\right)$$

$$\langle x'|x''\rangle = \int dp'\langle x'|p'\rangle\langle p'|x''\rangle$$

$$\begin{aligned}\delta(x'-x'') &= |N|^2\int dp'\exp\left[\frac{ip'(x'-x'')}{\hbar}\right]\\ &= 2\pi\hbar|N|^2\delta(x'-x'')\end{aligned}$$

$$\langle x'|\alpha\rangle = \int dp' \langle x'|p'\rangle\langle p'|\alpha\rangle$$

$$\langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$$

$$\psi_\alpha(x') = \left[\frac{1}{\sqrt{2\pi\hbar}}\right]\int dp' \exp\left(\frac{ip'x'}{\hbar}\right)\phi_\alpha(p')$$

$$\phi_\alpha(p') = \left[\frac{1}{\sqrt{2\pi\hbar}}\right]\int dx' \exp\left(\frac{-ip'x'}{\hbar}\right)\psi_\alpha(x')$$

Gaussian Wave Packets

It is instructive to look at a physical example to illustrate our basic formalism.

We consider what is known as a Gaussian wave packet, whose x-space wave function is given by

$$\langle x'|\alpha\rangle = \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[ikx' - \frac{x'^2}{2d^2}\right]$$

$$\langle x\rangle = \int_{-\infty}^{\infty} dx'\langle\alpha|x'\rangle x'\langle x'|\alpha\rangle = \int_{-\infty}^{\infty} dx'|\langle x'|\alpha\rangle|^2 x' = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx'\, x'^2 |\langle x'|\alpha\rangle|^2$$

$$= \left(\frac{1}{\sqrt{\pi}\,\mathrm{d}}\right)\int_{-\infty}^{\infty} dx'\, x'^2 \exp\left[\frac{-x'^2}{d^2}\right]$$

$$\left\langle (\Delta x)^2 \right\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{d^2}{2}$$

$$= \frac{d^2}{2}$$

$$\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle = \frac{\hbar^2}{4}$$

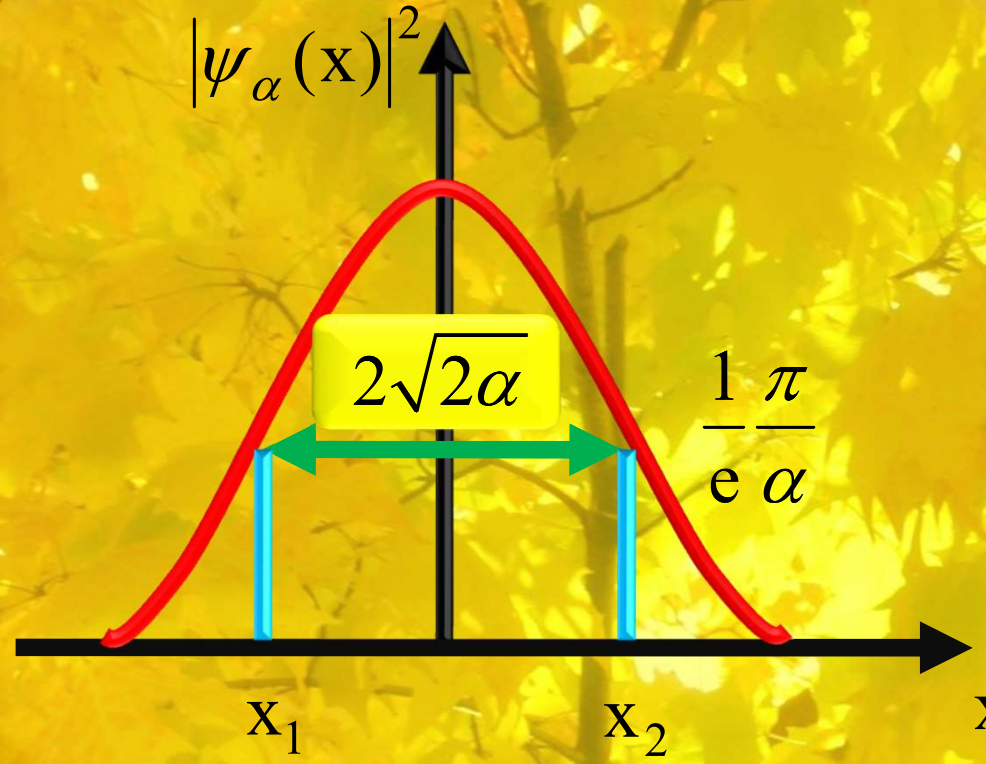
$$\langle p\rangle = \hbar k$$

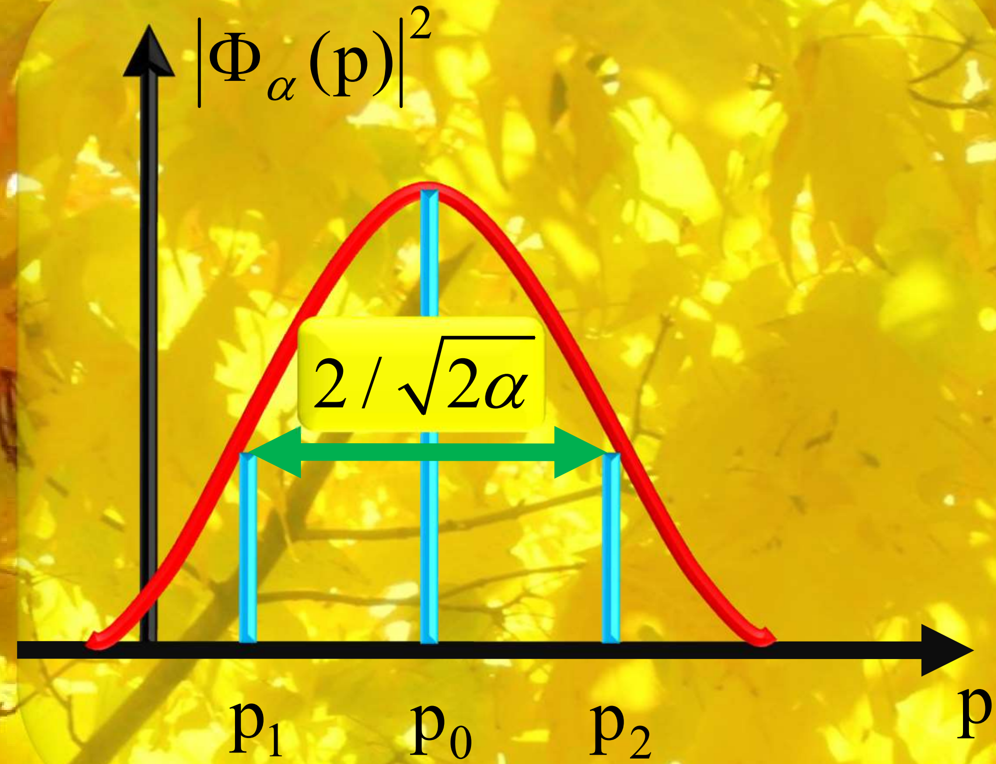
$$\langle p'|\alpha\rangle = \left(\frac{1}{\sqrt{2\pi\hbar}}\right)\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\int_{-\infty}^{\infty} dx' \exp\left(\frac{-ip'x'}{\hbar} + ikx' - \frac{x'^2}{2d^2}\right)$$

$$\langle p^2\rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

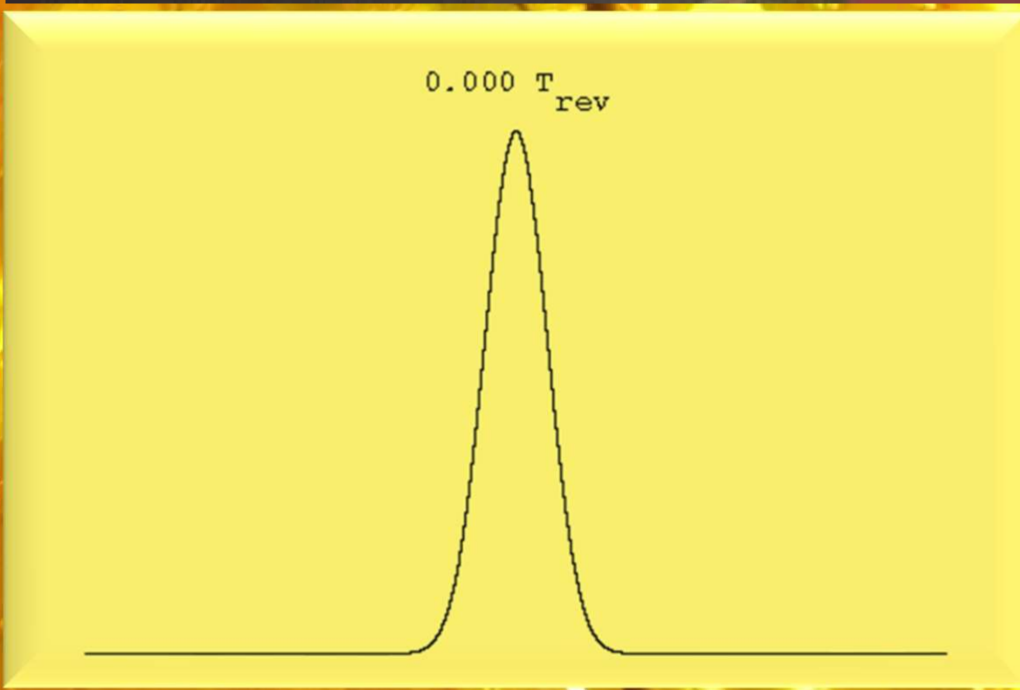
$$= \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\,\exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right]$$

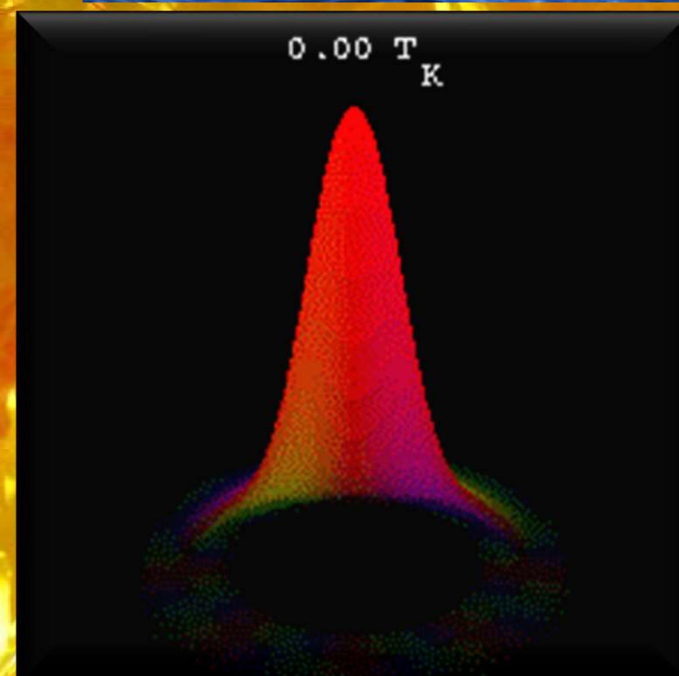
$$\left\langle (\Delta p)^2 \right\rangle = \langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar^2}{2d^2}$$





As an extreme example, suppose we let d->∞. The position-space wave function then becomes a plane wave extending over all space; the probability of finding the particle is just constant, independent of x'. In contrast, the momentum-space wave function is δ-function-like and is sharply peaked at ħk. In the opposite extreme, by letting d -> 0, we obtain a position-space wave function localized like the δ-function, but the momentum-space wave function is just constant, independent of p'.


0.000 T rev


0.00 T K

Generalization to Three Dimensions

$$\mathbf{x}|\mathbf{x}'\rangle = \mathbf{x}'|\mathbf{x}'\rangle$$

$$\langle\mathbf{x}'|\mathbf{x}''\rangle = \delta^3(\mathbf{x}'-\mathbf{x}'')$$

$$\mathbf{p}|\mathbf{p}'\rangle = \mathbf{p}'|\mathbf{p}'\rangle$$

$$\langle\mathbf{p}'|\mathbf{p}''\rangle = \delta^3(\mathbf{p}'-\mathbf{p}'')$$

$$\delta^3(\mathbf{x}'-\mathbf{x}'') = \delta(x'-x'')\delta(y'-y'')\delta(z'-z'')$$

$$\int d^3x'|\mathbf{x}'\rangle\langle\mathbf{x}'| = 1$$

$$|\alpha\rangle = \int d^3x'|\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle$$

$$\int d^3p'|\mathbf{p}'\rangle\langle\mathbf{p}'| = 1$$

$$|\alpha\rangle = \int d^3p'|\mathbf{p}'\rangle\langle\mathbf{p}'|\alpha\rangle$$

$$\phi_\alpha(\mathbf{p}') = \left[\frac{1}{(2\pi\hbar)^{3/2}}\right] \int d^3x' \exp\left(\frac{-i\mathbf{p}'\cdot\mathbf{x}'}{\hbar}\right) \psi_\alpha(\mathbf{x}')$$

$$\langle \beta|\mathbf{p}|\alpha\rangle = \int d^3x' \psi_\beta^*(\mathbf{x}')(-i\hbar\nabla)\psi_\alpha(\mathbf{x}')$$

$$\langle \mathbf{x}'|\mathbf{p}'\rangle = \left[\frac{1}{(2\pi\hbar)^{3/2}}\right] \exp\left(\frac{i\mathbf{p}'\cdot\mathbf{x}'}{\hbar}\right)$$

$$\psi_\alpha(\mathbf{x}') = \left[\frac{1}{(2\pi\hbar)^{3/2}}\right] \int d^3p' \exp\left(\frac{i\mathbf{p}'\cdot\mathbf{x}'}{\hbar}\right) \phi_\alpha(\mathbf{p}')$$