POTENTIALS AND GAUGE TRANSFORMATIONS

Constant Potentials

In classical mechanics it is well known that the zero point of the potential energy is of no physical significance. The time development of dynamic variables such as $\mathbf{x}(t)$ and $\mathbf{L}(t)$ is independent of whether we use $V(\mathbf{x})$ or $V(\mathbf{x})+V_0$ with V_0 constant both in space and time. The force that appears in Newton's second law depends only on the gradient of the potential; an additive constant is clearly irrelevant. What is the analogous situation in quantum mechanics?

We look at the time evolution of a Schrödinger-picture state ket subject to some potential. Let $|\alpha, t_0; t\rangle$ be a state ket in the presence of $V(\mathbf{x})$, and $|\alpha, t_0; t\rangle$, the corresponding state ket appropriate for

$$\tilde{V}(\mathbf{x}) = V(\mathbf{x}) + V_0$$

$$\widetilde{|\alpha, t_0; t\rangle} = \exp\left[-i\left(\frac{\mathbf{p}^2}{2m} + V(x) + V_0\right)\frac{(t-t_0)}{\hbar}\right]|\alpha\rangle$$
$$= \exp\left[\frac{-iV_0(t-t_0)}{\hbar}\right]|\alpha, t_0; t\rangle.$$

In other words, the ket computed under the influence of $\tilde{V}$ has a time dependence different only by a phase factor $\exp[-iV_0(t-t_0)/\hbar]$. For stationary states, this means that if the time dependence computed with $V(\mathbf{x})$ is $\exp[-iE(t-t_0)/\hbar]$, then the corresponding time dependence computed with $V(\mathbf{x})+V_0$ is $\exp[-i(E+V_0)(t-t_0)/\hbar]$. In other words, the use of $\tilde{V}$ in place of V just amounts to the following change:

$$E \rightarrow E + V_0$$

which the reader probably guessed immediately. Observable effects such as the time evolution of expectation values of $\langle \mathbf{x} \rangle$ and $\langle \mathbf{S} \rangle$ always depend on energy *differences* [see (2.1.47)]; the Bohr frequencies that characterize the sinusoidal time dependence of expectation values are the same whether we use $V(\mathbf{x})$ or $V(\mathbf{x})+V_0$. In general, there can be no difference in the expectation values of observables if every state ket in the world is multiplied by a common factor $\exp[-iV_0(t-t_0)/\hbar]$.

Trivial as it may seem, we see here the first example of a class of transformations known as **gauge transformations**. The change in our convention for the zero-point energy of the potential

$$V(\mathbf{x}) \to V(\mathbf{x}) + V_0$$

must be accompanied by a change in the state ket

$$|\alpha, t_0; t\rangle \to \exp\left[\frac{-iV_0(t-t_0)}{\hbar}\right]|\alpha, t_0; t\rangle$$

Of course, this change implies the following change in the wave function:

$$\psi(\mathbf{x}', t) \to \exp\left[\frac{-iV_0(t-t_0)}{\hbar}\right]\psi(\mathbf{x}', t)$$

Next we consider V_0 that is spatially uniform but dependent on time. We then easily see that the analogue of (2.6.5) is

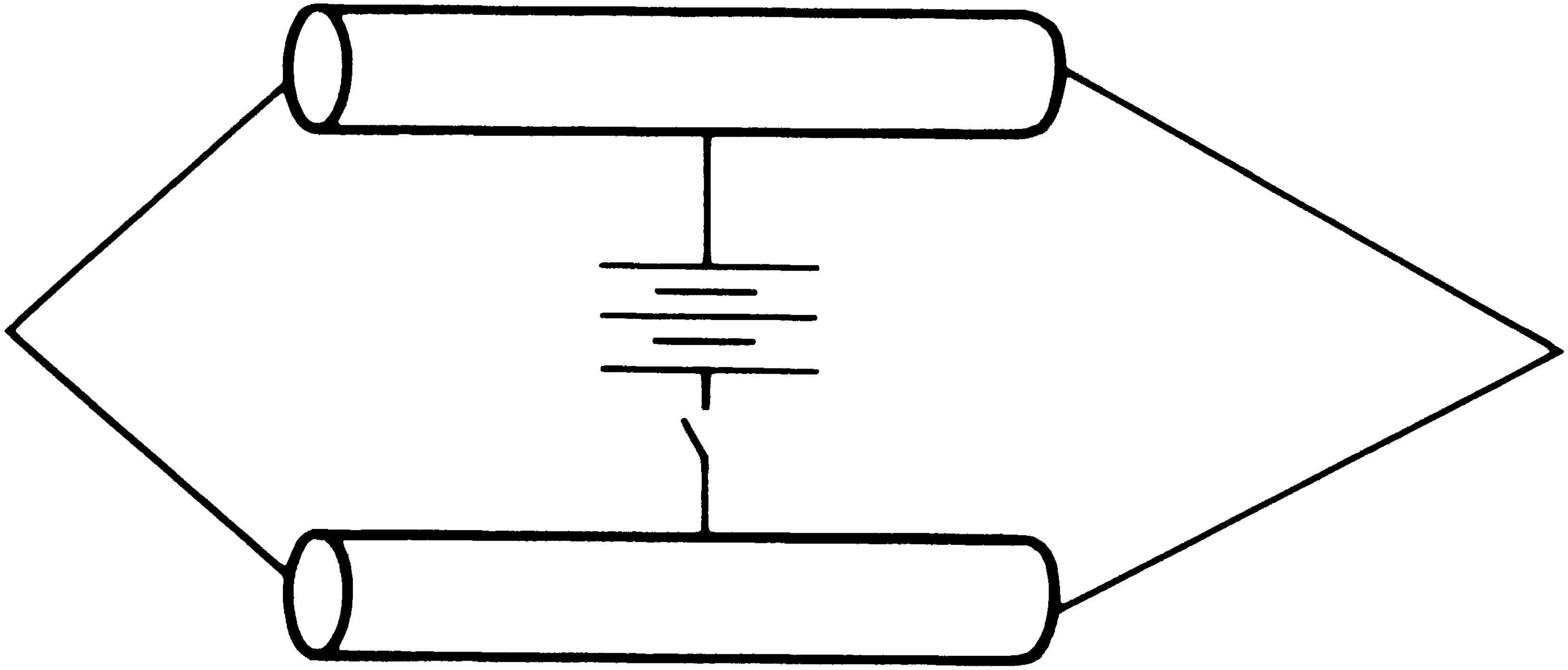
$$|\alpha, t_0; t\rangle \to \exp\left[-i\int_{t_0}^{t} dt' \frac{V_0(t')}{\hbar}\right]|\alpha, t_0; t\rangle$$

Physically, the use of $V(\mathbf{x}) + V_0(t)$ in place of $V(\mathbf{x})$ simply means that we are choosing a new zero point of the energy scale at each instant of time.

Even though the choice of the absolute scale of the potential is arbitrary, *potential differences* are of nontrivial physical significance and, in fact, can be detected in a very striking way. To illustrate this point, let us consider the arrangement shown in Figure 2.4. A beam of charged particles is split into two parts, each of which enters a metallic cage. If we so desire, we can maintain a finite potential difference between the two cages by turning on a switch, as shown. A particle in the beam can be visualized as a wave packet whose dimension is much smaller than the dimension of the cage. Suppose we switch on the potential difference only after the wave packets enter the cages and switch it off before the wave packets leave the cages. The particle in the cage experiences *no force* because inside the cage the potential is spatially uniform; hence no electric field is present. Now let us recombine the two beam components in such a way that they meet in the interference region of Figure 2.4. Because of the existence of the potential, each beam component suffers a phase change, as indicated by (2.6.7). As a result, there is an observable interference term in the beam intensity in the interference region, namely,

$$\cos(\phi_1 - \phi_2), \qquad \sin(\phi_1 - \phi_2)$$

$$\phi_1 - \phi_2 = \left(\frac{1}{\hbar}\right) \int_{t_i}^{t_f} dt \, [V_2(t) - V_1(t)]$$



So despite the fact that the particle experiences no force, there is an observable effect that depends on whether $V_2(t) - V_1(t)$ has been applied. Notice that this effect is *purely quantum mechanical*; in the limit $\hbar \to 0$, the interesting interference effect gets washed out because the oscillation of the cosine becomes infinitely rapid.*

Gravity in Quantum Mechanics

There is an experiment that exhibits in a striking manner how a gravitational effect appears in quantum mechanics. Before describing it, we first comment on the role of gravity in both classical and quantum mechanics.

Consider the classical equation of motion for a purely falling body:

$$m\ddot{\mathbf{x}} = -m\nabla\Phi_{\text{grav}} = -mg\hat{\mathbf{z}}.$$

The mass term drops out; so in the absence of air resistance, a feather and a stone would behave in the same way—à la Galileo—under the influence of gravity. This is, of course, a direct consequence of the equality of the gravitational and the inertial masses. Because the mass does not appear in the equation of a particle trajectory, gravity in classical mechanics is often said to be a purely geometric theory.

The situation is rather different in quantum mechanics. In the wave-mechanical formulation, the analogue of (2.6.10) is

$$\left[-\left(\frac{\hbar^2}{2m}\right)\nabla^2 + m\Phi_{\text{grav}}\right]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

The mass no longer cancels; instead it appears in the combination $\hbar/m$, so in a problem where $\hbar$ appears, m is also expected to appear. We can see this point also using the Feynman path-integral formulation of a falling body based on

$$\langle \mathbf{x}_n, t_n | \mathbf{x}_{n-1}, t_{n-1}\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\,\exp\left[i\int_{t_{n-1}}^{t_n} dt\,\frac{\left(\frac{1}{2}m\dot{\mathbf{x}}^2 - mgz\right)}{\hbar}\right] \qquad (t_n - t_{n-1} = \Delta t \to 0)$$

Here again we see that m appears in the combination $m/\hbar$. This is in sharp contrast with Hamilton's classical approach, based on

$$\delta\int_{t_1}^{t_2} dt\left(\frac{m\dot{\mathbf{x}}^2}{2} - mgz\right) = 0$$

where m can be eliminated in the very beginning.

Starting with the Schrödinger equation (2.6.11), we may derive the Ehrenfest theorem

$$\frac{d^2}{dt^2}\langle \mathbf{x} \rangle = -g\hat{\mathbf{z}}. \tag{2.6.14}$$

However, $\hbar$ does not appear here, nor does m. To see a *nontrivial* quantum-mechanical effect of gravity, we must study effects in which $\hbar$ appears explicitly—and consequently where we expect the mass to appear—in contrast with purely gravitational phenomena in classical mechanics.

Until 1975, there had been no direct experiment that established the presence of the $m\Phi_{\text{grav}}$ term in (2.6.11). To be sure, a free fall of an elementary particle had been observed, but the classical equation of motion —or the Ehrenfest theorem (2.6.14), where $\hbar$ does not appear—sufficed to account for this. The famous "weight of photon" experiment of V. Pound and collaborators did not test gravity in the quantum domain either because they measured a frequency shift where $\hbar$ does not explicitly appear.

On the microscopic scale, gravitational forces are too weak to be readily observable. To appreciate the difficulty involved in seeing gravity in bound-state problems, let us consider the ground state of an electron and a neutron bound by gravitational forces. This is the gravitational analogue of the hydrogen atom, where an electron and a proton are bound by Coulomb forces. At the same distance, the gravitational force between the electron and the neutron is weaker than the Coulomb force between the electron and the proton by a factor of $\sim 2 \times 10^{39}$. The Bohr radius involved here can be obtained simply:

$$a_0 = \frac{\hbar^2}{e^2 m_e} \rightarrow \frac{\hbar^2}{G_N m_e^2 m_n}, \tag{2.6.15}$$

where G_N is Newton's gravitational constant. If we substitute numbers in the equation, the Bohr radius of this gravitationally bound system turns out to be $\sim 10^{31}$ cm, or $\sim 10^{13}$ light years, which is larger than the estimated radius of the universe by a few orders of magnitude!

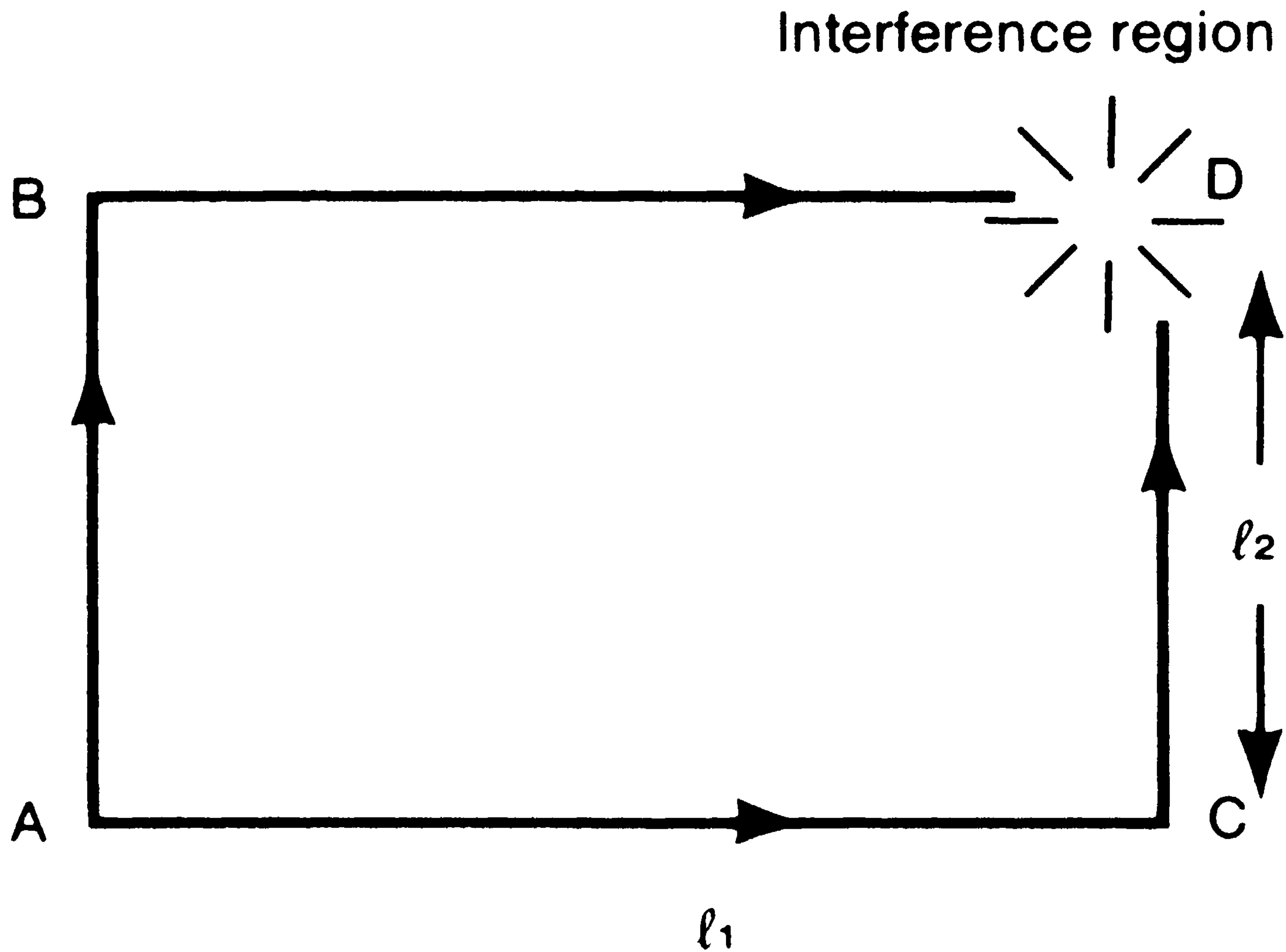
We now discuss a remarkable phenomenon known as **gravity-induced quantum interference**. A nearly monoenergetic beam of particles—in practice, thermal neutrons—is split into two parts and then brought together as shown in Figure 2.5. In actual experiments the neutron beam is split and bent by silicon crystals, but the details of this beautiful art of neutron interferometry do not concern us here. Because the size of the wave packet can be assumed to be much smaller than the macroscopic dimension of the loop formed by the two alternate paths, we can apply the concept of a classical trajectory. Let us first suppose that path $A \to B \to D$ and path $A \to C \to D$ lie in a horizontal plane. Because the absolute zero of the potential due to gravity is of no significance, we can set $V = 0$ for any phenomenon that takes place in this plane; in other words, it is legitimate to ignore gravity altogether. The situation is very different if the plane formed by the two alternate paths is rotated around segment AC by angle δ. This time the potential at level BD is higher than that at level AC by $mgl_2 \sin\delta$, which means that the state ket associated with path BD "rotates faster." This leads to a gravity-induced phase difference between the amplitudes for the two wave packets arriving at D. Actually there is also a gravity-induced phase change associated with AB and also with CD, but the effects cancel as we compare the two alternate paths. The net result is that the wave packet arriving at D via path ABD suffers a phase change

$$\exp\left[\frac{-im_n g l_2 \sin\delta\, T}{\hbar}\right] \tag{2.6.16}$$

relative to that of the wave packet arriving at D via path ACD, where T is the time spent for the wave packet to go from B to D (or from A to C) and m_n, the neutron mass. We can control this phase difference by rotating the plane of Figure 2.5; δ can change from 0 to $\pi/2$, or from 0 to $-\pi/2$. Expressing the time spent T, or $l_1/v_{\text{wave packet}}$, in terms of $\lambda\!\!\!^{-}$, the de Broglie wavelength of the neutron, we obtain the following expression for the phase difference:

$$\phi_{ABD} - \phi_{ACD} = -\frac{\left(m_n^2 g l_1 l_2 \lambda\!\!\!^{-} \sin\delta\right)}{\hbar^2}. \tag{2.6.17}$$

In this manner we predict an observable interference effect that depends on angle δ, which is reminiscent of fringes in Michelson-type interferometers in optics.

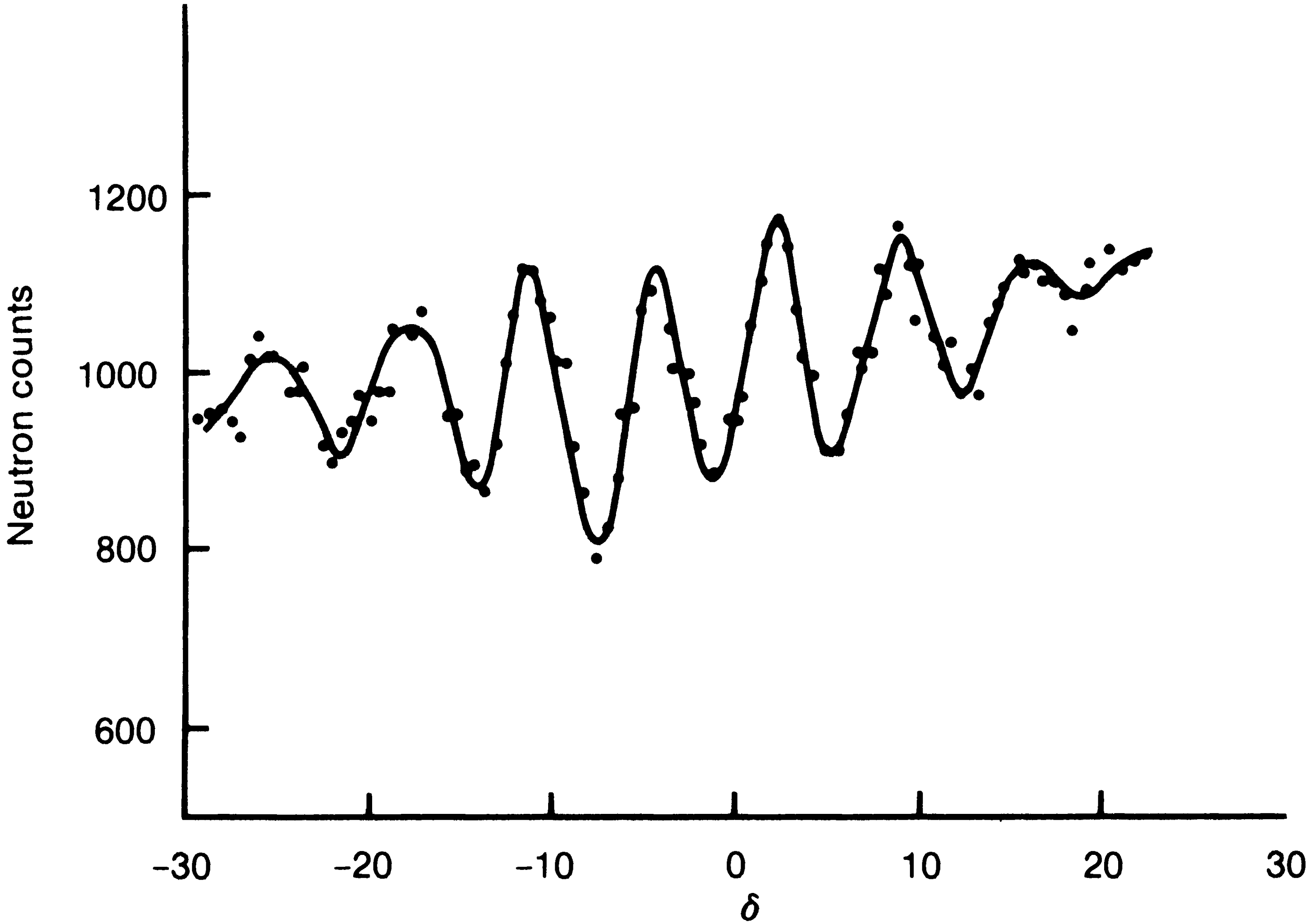

Interference region
B
D
ℓ_2
A
C
ℓ_1

An alternative, more wave-mechanical way to understand (2.6.17) follows. Because we are concerned with a time-independent potential, the sum of the kinetic energy and the potential energy is constant:

$$\frac{\mathbf{p}^2}{2m} + mgz = E. \tag{2.6.18}$$

The difference in height between level BD and level AC implies a slight difference in $\mathbf{p}$, or $\lambda\!\!\!^{-}$. As a result, there is an accumulation of phase differences due to the $\lambda\!\!\!^{-}$ difference. It is left as an exercise to show that this wave-mechanical approach also leads to result (2.6.17).

What is interesting about expression (2.6.17) is that its magnitude is neither too small nor too large; it is just right for this interesting effect to be detected with thermal neutrons traveling through paths of "table-top" dimensions. For $\lambda = 1.42$ Å (comparable to interatomic spacing in silicon) and $l_1 l_2 = 10\ \text{cm}^2$, we obtain 55.6 for $m_n^2 g l_1 l_2 \lambda\!\!\!^{-} / \hbar^2$. As we rotate the loop plane gradually by 90°, we predict the intensity in the interference region to exhibit a series of maxima and minima; quantitatively we should see $55.6/2\pi \simeq 9$ oscillations. It is extraordinary that such an effect has indeed been observed experimentally; see Figure 2.6 taken from a 1975 experiment of R. Colella, A. Overhauser, and S. A. Werner. The phase shift due to gravity is seen to be verified to well within 1%.



We emphasize that this effect is purely quantum mechanical because as $\hbar \to 0$, the interference pattern gets washed out. The gravitational potential has been shown to enter into the Schrödinger equation just as expected. This experiment also shows that gravity is not purely geometric at the quantum level because the effect depends on $(m/\hbar)^2$.*

Gauge Transformations in Electromagnetism

Let us now turn to potentials that appear in electromagnetism. We consider an electric and a magnetic field derivable from the time-independent scalar and vector potential, $\phi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$:

$$\mathbf{E} = -\nabla\phi, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

The Hamiltonian for a particle of electric charge e ($e < 0$ for the electron) subjected to the electromagnetic field is taken from classical physics to be

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 + e\phi.$$

In quantum mechanics ϕ and $\mathbf{A}$ are understood to be functions of the position *operator* $\mathbf{x}$ of the charged particle. Because $\mathbf{p}$ and $\mathbf{A}$ do not commute, some care is needed in interpreting (2.6.20). The safest procedure is to write

$$\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)^2 \to p^2-\left(\frac{e}{c}\right)(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p})+\left(\frac{e}{c}\right)^2\mathbf{A}^2.$$

In this form the Hamiltonian is obviously Hermitian.

To study the dynamics of a charged particle subjected to ϕ and $\mathbf{A}$, let us first proceed in the Heisenberg picture. We can evaluate the time derivative of $\mathbf{x}$ in a straightforward manner as

$$\mathbf{\Pi}\equiv m\frac{d\mathbf{x}}{dt}=\mathbf{p}-\frac{e\mathbf{A}}{c}.$$

Even though we have

$$[p_i, p_j]=0$$

for canonical momentum, the analogous commutator does not vanish for mechanical momentum. Instead we have

$$[\Pi_i, \Pi_j] = \left(\frac{i\hbar e}{c}\right)\varepsilon_{ijk} B_k,$$

as the reader may easily verify. Rewriting the Hamiltonian as

$$H = \frac{\boldsymbol{\Pi}^2}{2m} + e\phi$$

and using the fundamental commutation relation, we can derive the quantum-mechanical version of the Lorentz force, namely,

$$m\frac{d^2\mathbf{x}}{dt^2} = \frac{d\boldsymbol{\Pi}}{dt} = e\left[\mathbf{E} + \frac{1}{2c}\left(\frac{d\mathbf{x}}{dt} \times \mathbf{B} - \mathbf{B} \times \frac{d\mathbf{x}}{dt}\right)\right]$$

This then is Ehrenfest's theorem, written in the Heisenberg picture, for the charged particle in the presence of **E** and **B**.

We now study Schrödinger's wave equation with ϕ and $\mathbf{A}$. Our first task is to sandwich H between $\langle \mathbf{x}'|$ and $|\alpha, t_0; t\rangle$. The only term with which we have to be careful is

$$\begin{aligned}
&\langle \mathbf{x}'|\left[\mathbf{p} - \frac{e\mathbf{A}(\mathbf{x})}{c}\right]^2 |\alpha, t_0; t\rangle \\
&\qquad = \left[-i\hbar\nabla' - \frac{e\mathbf{A}(\mathbf{x}')}{c}\right]\langle \mathbf{x}'|\left[\mathbf{p} - \frac{e\mathbf{A}(\mathbf{x})}{c}\right]|\alpha, t_0; t\rangle \\
&\qquad = \left[-i\hbar\nabla' - \frac{e\mathbf{A}(\mathbf{x}')}{c}\right]\cdot\left[-i\hbar\nabla' - \frac{e\mathbf{A}(\mathbf{x}')}{c}\right]\langle \mathbf{x}'|\alpha, t_0; t\rangle.
\end{aligned}$$

It is important to emphasize that the first ∇' in the last line can differentiate *both* $\langle \mathbf{x}'|\alpha, t_0; t\rangle$ *and* $\mathbf{A}(\mathbf{x}')$. Combining everything, we have

$$\begin{aligned}
&\frac{1}{2m}\left[-i\hbar\nabla' - \frac{e\mathbf{A}(\mathbf{x}')}{c}\right]\cdot\left[-i\hbar\nabla' - \frac{e\mathbf{A}(\mathbf{x}')}{c}\right]\langle \mathbf{x}'|\alpha, t_0; t\rangle \\
&\qquad + e\phi(\mathbf{x}')\langle \mathbf{x}'|\alpha, t_0; t\rangle = i\hbar\frac{\partial}{\partial t}\langle \mathbf{x}'|\alpha, t_0; t\rangle.
\end{aligned}$$

From this expression we readily obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla' \cdot \mathbf{j} = 0,$$

where ρ is $|\psi|^2$ as before, with $\langle \mathbf{x}'|\alpha, t_0; t\rangle$ written as ψ, but for the probability flux $\mathbf{j}$ we have

$$\mathbf{j} = \left(\frac{\hbar}{m}\right)\mathrm{Im}(\psi^* \nabla' \psi) - \left(\frac{e}{mc}\right)\mathbf{A}|\psi|^2,$$

which is just what we expect from the substitution

$$\nabla' \to \nabla' - \left(\frac{ie}{\hbar c}\right)\mathbf{A}.$$

Writing the wave function of $\sqrt{\rho}\,\exp(iS/\hbar)$ [see (2.4.18)], we obtain an alternative form for $\mathbf{j}$, namely,

$$\mathbf{j} = \left(\frac{\rho}{m}\right)\left(\nabla S - \frac{e\mathbf{A}}{c}\right),$$

which is to be compared with (2.4.20). We will find this form to be convenient in discussing superconductivity, flux quantization, and so on. We also note that the space integral of $\mathbf{j}$ is the expectation value of kinematical momentum (not canonical momentum) apart from $1/m$:

$$\int d^3x'\mathbf{j} = \frac{\langle \mathbf{p} - e\mathbf{A}/c \rangle}{m} = \langle \mathbf{\Pi} \rangle / m.$$

We are now in a position to discuss the subject of **gauge transformations** in electromagnetism. First, consider

$$\phi \to \phi + \lambda, \qquad \mathbf{A} \to \mathbf{A},$$

with λ constant, that is, independent of $\mathbf{x}$ and t. Both $\mathbf{E}$ and $\mathbf{B}$ obviously remain unchanged. This transformation just amounts to a change in the zero point of the energy scale, a possibility treated in the beginning of this section; we just replace V by $e\phi$. We have already discussed the accompanying change needed for the state ket [see (2.6.5)], so we do not dwell on this transformation any further.

Much more interesting is the transformation

$$\phi \rightarrow \phi, \qquad \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda,$$

where Λ is a function of $\mathbf{x}$. The static electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are unchanged under (2.6.36). Both (2.6.35) and (2.6.36) are special cases of

$$\phi \rightarrow \phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t}, \qquad \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda,$$

which leave $\mathbf{E}$ and $\mathbf{B}$, given by

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A},$$

unchanged, but in the following we do not consider time-dependent fields and potentials. In the remaining part of this section the term *gauge transformation* refers to (2.6.36).

In classical physics observable effects such as the trajectory of a charged particle are independent of the gauge used, that is, of the particular choice of Λ we happen to adopt. Consider a charged particle in a uniform magnetic field in the z-direction

$$\mathbf{B} = B\hat{\mathbf{z}}.$$

This magnetic field may be derived from

$$A_x = \frac{-By}{2}, \qquad A_y = \frac{Bx}{2}, \qquad A_z = 0$$

or also from

$$A_x = -By, \qquad A_y = 0, \qquad A_z = 0.$$

The second form is obtained from the first by

$$\mathbf{A} \to \mathbf{A} - \nabla\left(\frac{Bxy}{2}\right),$$

which is indeed of the form of (2.6.36). Regardless of which **A** we may use, the trajectory of the charged particle with a given set of initial conditions is the same; it is just a helix—a uniform circular motion when projected in the xy-plane, superposed with a uniform rectilinear motion in the z-direction. Yet if we look at p_x and p_y, the results are very different. For one thing, p_x is a constant of the motion when (2.6.41) is used but not when (2.6.40) is used.

Recall Hamilton's equations of motion:

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x}, \qquad \frac{dp_y}{dt} = -\frac{\partial H}{\partial y}, \ldots .$$

In general, the canonical momentum $\mathbf{p}$ is *not* a gauge-invariant quantity; its numerical value depends on the particular gauge used even when we are referring to the same physical situation. In contrast, the *kinematic* momentum $\mathbf{\Pi}$, or $m d\mathbf{x}/dt$, that traces the trajectory of the particle *is* a gauge-invariant quantity, as one may explicitly verify. Because $\mathbf{p}$ and $m d\mathbf{x}/dt$ are related via (2.6.23), $\mathbf{p}$ must change to compensate for the change in $\mathbf{A}$ given by (2.6.42).

We now return to quantum mechanics. We believe that it is reasonable to demand that the expectation values in quantum mechanics behave in a manner similar to the corresponding classical quantities under gauge transformations, so $\langle \mathbf{x} \rangle$ and $\langle \mathbf{\Pi} \rangle$ are *not* to change under gauge transformations, while $\langle \mathbf{p} \rangle$ is expected to change.

Let us denote by $|\alpha\rangle$ the state ket in the presence of $\mathbf{A}$; the state ket for the same physical situation when

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\Lambda$$

is used in place of $\mathbf{A}$ is denoted by $|\tilde{\alpha}\rangle$. Here Λ, as well as $\mathbf{A}$, is a function of the position operator $\mathbf{x}$. Our basic requirements are

$$\langle\alpha|\mathbf{x}|\alpha\rangle = \langle\tilde{\alpha}|\mathbf{x}|\tilde{\alpha}\rangle$$

and

$$\langle\alpha|\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)|\alpha\rangle = \langle\tilde{\alpha}|\left(\mathbf{p} - \frac{e\tilde{\mathbf{A}}}{c}\right)|\tilde{\alpha}\rangle.$$

In addition, we require, as usual, the norm of the state ket to be preserved:

$$\langle\alpha|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\alpha}\rangle.$$

We must construct an operator $\mathcal{G}$ that relates $|\tilde{\alpha}\rangle$ to $|\alpha\rangle$:

$$|\tilde{\alpha}\rangle = \mathcal{G}|\alpha\rangle.$$

Invariance properties (2.6.45a) and (2.6.45b) are guaranteed if

$$\mathscr{G}^{\dagger}\mathbf{x}\mathscr{G} = \mathbf{x}$$

and

$$\mathscr{G}^{\dagger}\left(\mathbf{p} - \frac{e\mathbf{A}}{c} - \frac{e\nabla\Lambda}{c}\right)\mathscr{G} = \mathbf{p} - \frac{e\mathbf{A}}{c}.$$

We assert that

$$\mathscr{G} = \exp\left[\frac{ie\Lambda(\mathbf{x})}{\hbar c}\right]$$

will do the job. First, $\mathscr{G}$ is unitary, so (2.6.46) is all right. Second, (2.6.48a) is obviously satisfied because $\mathbf{x}$ commutes with any function of $\mathbf{x}$. As for (2.6.48b), just note that

$$\begin{aligned}\exp\left(\frac{-ie\Lambda}{\hbar c}\right)\mathbf{p}\exp\left(\frac{ie\Lambda}{\hbar c}\right) &= \exp\left(\frac{-ie\Lambda}{\hbar c}\right)\left[\mathbf{p}, \exp\left(\frac{ie\Lambda}{\hbar c}\right)\right] + \mathbf{p}\\ &= -\exp\left(\frac{-ie\Lambda}{\hbar c}\right)i\hbar\nabla\left[\exp\left(\frac{ie\Lambda}{\hbar c}\right)\right] + \mathbf{p} = \mathbf{p} + \frac{e\nabla\Lambda}{c}\end{aligned}$$

The invariance of quantum mechanics under gauge transformations can also be demonstrated by looking directly at the Schrödinger equation. Let $|\alpha, t_0; t\rangle$ be a solution to the Schrödinger equation in the presence of $\mathbf{A}$:

$$\left[\frac{(\mathbf{p}-e\mathbf{A}/c)^2}{2m}+e\phi\right]|\alpha, t_0; t\rangle = i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle.$$

The corresponding solution in the presence of $\tilde{\mathbf{A}}$ must satisfy

$$\left[\frac{(\mathbf{p}-e\mathbf{A}/c-e\nabla\Lambda/c)^2}{2m}+e\phi\right]\widetilde{|\alpha, t_0; t\rangle} = i\hbar\frac{\partial}{\partial t}\widetilde{|\alpha, t_0; t\rangle}.$$

We see that if the new ket is taken to be

$$\widetilde{|\alpha, t_0; t\rangle} = \exp\left(\frac{ie\Lambda}{\hbar c}\right)|\alpha, t_0; t\rangle$$

in accordance with (2.6.49), then the new Schrödinger equation (2.6.52) will be satisfied; all we have to note is that

$$\exp\left(\frac{-ie\Lambda}{\hbar c}\right)\left(\mathbf{p}-\frac{e\mathbf{A}}{c}-\frac{e\nabla\Lambda}{c}\right)^2\exp\left(\frac{ie\Lambda}{\hbar c}\right)=\left(\mathbf{p}-\frac{e\mathbf{A}}{c}\right)^2, \quad (2.6.54)$$

which follows from applying (2.6.50) twice.

Equation (2.6.53) also implies that the corresponding wave equations are related via

$$\tilde{\psi}(\mathbf{x}', t) = \exp\left[\frac{ie\Lambda(\mathbf{x}')}{\hbar c}\right]\psi(\mathbf{x}', t), \tag{2.6.55}$$

where $\Lambda(\mathbf{x}')$ is now a real function of the position vector eigenvalue $\mathbf{x}'$. This can, of course, be verified also by directly substituting (2.6.55) into Schrödinger's wave equation with $\mathbf{A}$ replaced by $\mathbf{A} + \nabla\Lambda$. In terms of ρ and S, we see that ρ is unchanged but S is modified as follows:

$$S \to S + \frac{e\Lambda}{c}. \tag{2.6.56}$$

This is highly satisfactory because we see that the probability flux given by (2.6.33) is then gauge invariant.

To summarize, when vector potentials in different gauges are used for the same physical situation, the corresponding state kets (or wave functions) must necessarily be different. However, only a simple change is needed; we can go from a gauge specified by $\mathbf{A}$ to another specified by $\mathbf{A} + \nabla\Lambda$ by merely multiplying the old ket (the old wave function) by $\exp[ie\Lambda(\mathbf{x})/\hbar c]$ ($\exp[ie\Lambda(\mathbf{x}')/\hbar c]$). The **canonical momentum**, defined as the generator of translation, is manifestly *gauge dependent* in the sense that its expectation value depends on the particular gauge chosen, while the **kinematic momentum** and the probability flux are *gauge invariant*.

The reader may wonder why invariance under (2.6.49) is called *gauge invariance*. This word is the translation of the German *Eichinvarianz*, where *Eich* means *gauge*. There is a historical anecdote that goes with the origin of this term.

Consider some function of position at $\mathbf{x}$: $F(\mathbf{x})$. At a neighboring point we obviously have

$$F(\mathbf{x}+d\mathbf{x}) \simeq F(\mathbf{x})+(\nabla F)\cdot d\mathbf{x}. \tag{2.6.57}$$

But suppose we apply a scale change as we go from $\mathbf{x}$ to $\mathbf{x}+d\mathbf{x}$ as follows:

$$1\big|_{\text{at }\mathbf{x}} \to [1+\Sigma(\mathbf{x})\cdot d\mathbf{x}]\big|_{\text{at }\mathbf{x}+d\mathbf{x}}. \tag{2.6.58}$$

We must then rescale $F(\mathbf{x})$ as follows:

$$F(\mathbf{x}+d\mathbf{x})\big|_{\text{rescaled}} \simeq F(\mathbf{x})+[(\nabla+\Sigma)F]\cdot d\mathbf{x} \tag{2.6.59}$$

instead of (2.6.57). The combination $\nabla+\Sigma$ is similar to the gauge-invariant combination

$$\nabla-\left(\frac{ie}{\hbar c}\right)\mathbf{A} \tag{2.6.60}$$

encountered in (2.6.32) except for the absence of i. Historically, H. Weyl unsuccessfully attempted to construct a geometric theory of electromagnetism based on *Eichinvarianz* by identifying the scale function $\Sigma(\mathbf{x})$ in

(2.6.58) and (2.6.59) with the vector potential **A** itself. With the birth of quantum mechanics, V. Fock and F. London realized the importance of the gauge-invariant combination (2.6.60), and they recalled Weyl's earlier work by comparing Σ with i times **A**. We are stuck with the term *gauge invariance* even though the quantum-mechanical analogue of (2.6.58)

$$1\Big|_{\text{at }\mathbf{x}} \rightarrow \left[1-\left(\frac{ie}{\hbar c}\right)\mathbf{A}\cdot d\mathbf{x}\right]\Big|_{\text{at }\mathbf{x}+d\mathbf{x}} \tag{2.6.61}$$

would actually correspond to "phase change" rather than to "scale change."

Gauge Transformations in Electromagnetism

معادلات ماکسول در واحد گوسی در خلاء

به صورت زیراند:

$$\nabla . \mathbf{B} = 0$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) + \frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = 0$$

$$\nabla . \mathbf{E}(\mathbf{r}, t) = 4\pi\rho(\mathbf{r}, t)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) - \frac{1}{c}\frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} = \frac{4\pi}{c} j(\mathbf{r}, t)$$

که در آنها $\rho(r,t)$ و $j(r,t)$ چگالهای بار و جریانند که چشمه ها یا منابع تولید میدانهای الکترومغناطیسی

قانون یا معادله بقای بار (معادله پیوستگی):

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla.\mathbf{j}(\mathbf{r},t) = 0$$

به طور خودکار ارضا می‌شود، زیرا.

$$\nabla.4 \Rightarrow \overbrace{\nabla.[\nabla\times\mathbf{B}(\mathbf{r},t)]}^{0} - \frac{1}{c}\frac{\partial \overbrace{\nabla.\mathbf{E}(\mathbf{r},t)}_{4\pi\rho}}{\partial t} = \frac{4\pi}{c}\nabla.\mathbf{j}(\mathbf{r},t)$$

می توانیم دو معادله اول را با بیان میدانها بر حسب یک پتانسیل اسکالر $\Phi(r,t)$ ویک پتانسیل برداری $A(r,t)$ برآورده سازیم :

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} - \nabla \Phi(\mathbf{r},t)$$

زی

$$\nabla.(\nabla \times \mathbf{F}) = 0$$

ب

$$\nabla.\mathbf{B}(\mathbf{r},t) = \nabla.(\nabla \times \mathbf{A}(\mathbf{r},t)) = 0$$

زی

را:

$$\nabla \times [\nabla \Phi(\mathbf{r},t)] = 0$$

ب

$$\nabla \times \mathbf{E}(\mathbf{r},t) + \frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = 0$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\overset{\mathbf{B}}{\nabla \times \mathbf{A}(\mathbf{r},t)} - \overset{0}{\nabla \times [\nabla \Phi(\mathbf{r},t)]}$$

میدانهای الکتریکی و مغناطیسی پتانسیلهای اسکالر و برداری را به طور منحصر به

مثلا از تعریف پتانسیلهای

جدیدی به صورت زیر:

$$\mathbf{A}'(\mathbf{r},t)=\mathbf{A}(\mathbf{r},t)-\nabla f(\mathbf{r},t)$$

$$\Phi'(\mathbf{r},t)=\Phi(\mathbf{r},t)+\frac{1}{c}\frac{\partial f(\mathbf{r},t)}{\partial t}$$

نیز به راحتی دیده می شود که همان میدانهای الکتریکی و مغناطیسی را (همانند آنچه که پتانسیلهای برداری و اسکالر قدیمی انجام می دادند) به دست می دهند.

$$(\mathbf{A}\quad \Phi)\xrightarrow{\text{gauge transformation}}(\mathbf{A}'\quad \Phi')$$

تبدیل از مجموعه پتانسیلهای قدیم به پتانسیلهای جدید را یک تبدیل پیمانه ای

یعنی ناوردایی میدانهای الکتریکی و مغناطیسی اجازه می دهد تا بتوانیم تابع اختیاری f را به بهترین وجه

اگر توزیع بار استاتیک باشد یعنی چگالی بار مستقل از زمان باشد مناسب

$$\nabla . \mathbf{A} = 0$$

این پیمانه پیمانه کولمب نام دارد و در واقع یک نوع انتخاب

در این صورت

$$-\nabla^2 \Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla . \mathbf{A}(\mathbf{r},t)\right) = 4\pi\rho(\mathbf{r},t)$$

چون طرف راست مستقل از زمان است پس طرف چپ هم باید

$$-\nabla^2 \Phi(\mathbf{r},\cancel{t}) = 4\pi\rho(\mathbf{r})$$

$$-\nabla^2 \Phi(\mathbf{r}) = 4\pi\rho(\mathbf{r})$$

یعنی پتانسیل سکالر نیز مستقل از

با استفاده از نکته اخیر می توان معادله پتانسیل برداری را

$$-\nabla^2 \mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} + \nabla\left(\cancelto{0}{\nabla.\mathbf{A}} + \cancelto{0}{\frac{1}{c}\frac{\partial \Phi}{\partial t}}\right) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r},t)$$

که این چیزی جزء یک معادله موج کلاسیک در حضور

$$-\nabla^2 \mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} = \frac{4\pi}{c}\mathbf{j}(\mathbf{r},t)$$

اگر توزیع بار ایستا نباشد یعنی چگالی بار وابسته به زمان باشد، از پیمانه ای

این پیمانه سبب می شود که معادله پتانسیل برداری بدون تغییر باقی نماند، اما اکنون معادله پتانسیل اسکالری هم

$$\nabla.\mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$$

$$-\nabla^2 \mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} + \nabla\cancelto{0}{\left(\nabla.\mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right)} = \frac{4\pi}{c}\mathbf{j}(\mathbf{r},t)$$

بنابراین پیمانه لورنتس سبب می شود که پتانسیل

$$-\nabla^2\mathbf{A}(\mathbf{r},t)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}(\mathbf{r},t)}{\partial t^2}=\frac{4\pi}{c}\mathbf{j}(\mathbf{r},t)$$

که همان معادله سابق موج کلاسیکی پتانسیل برداری در پیمانه کولمب است.

$$-\nabla^2\Phi(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\big(\nabla.\mathbf{A}(\mathbf{r},t)\big)=4\pi\rho(\mathbf{r},t)$$

$$\nabla.\mathbf{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0$$

$$-\nabla^2\Phi(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\Phi}{\partial t}\right)=4\pi\rho(\mathbf{r},t)$$

$$-\nabla^2\Phi(\mathbf{r},t)+\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=4\pi\rho(\mathbf{r},t)$$

پس پتانسیل اسکالری هم همچون پتانسیل برداری در معادله موج کلاسیک صدق می کند.

معادله توصیف کننده برهمکنش یک الکترون نقطه ای به جرم μ با یک میدان الکترومغناطیسی عبارت از معادله نیروی لورنس کلاسیکی است:

$$\mu \frac{d^2\mathbf{r}}{dt^2} = -e\left[\mathbf{E}(\mathbf{r},t) + \frac{\mathbf{V}}{c} \times \mathbf{B}(\mathbf{r},t)\right]$$

حال بیان می کنیم که معادله حرکت کلاسیکی فوق به دست خواهد آمد اگر هامیلتونی کلاسیکی برای یک

$$H_0 = \frac{P^2}{2\mu}$$

با تغییر متغیر زیر

$$\mathbf{P} \longrightarrow \mathbf{P} + \frac{e}{c}\mathbf{A}(\mathbf{r}, t)$$

و پتانسیل eφ (که در اینجا فقط به پتانسیلهای اسکالر استاتیک علاقه مند هستیم) به آن اضافه شود، به طوری که:

$$H = \frac{1}{2\mu}\left[\mathbf{P}(\mathbf{r}, t) + \frac{e}{c}\mathbf{A}(\mathbf{r}, t)\right]^2 + e\Phi(\mathbf{r})$$

می توان (با استفاده از معادلات حرکت هامیلتون زیر) ثابت کرد که از هامیلتونی فوق می توان معادله

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x}$$

معادله شرودینگر متناظر با بردن پتانسیل استاتیک به سمت

$$\frac{1}{2\mu}\left[\mathbf{P}(\mathbf{r},t)+\frac{e}{c}\mathbf{A}(\mathbf{r},t)\right]^2\Psi(\mathbf{r},t)=\left(E+e\Phi(\mathbf{r})\right)\Psi(\mathbf{r},t)$$

طرف چپ عبارت

$$\frac{1}{2\mu}\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}(\mathbf{r},t)\right]\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}(\mathbf{r},t)\right]\Psi(\mathbf{r},t)$$

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi-\frac{ie\hbar}{\mu c}\mathbf{A}.\nabla\Psi-\frac{ie\hbar}{2\mu c}\overset{0}{(\nabla.\mathbf{A})}\Psi+\frac{e^2}{2\mu c^2}A^2\Psi$$

$$\boxed{-\frac{\hbar^2}{2\mu}\nabla^2\Psi-\frac{ie\hbar}{\mu c}\mathbf{A}.\nabla\Psi+\frac{e^2}{2\mu c^2}A^2\Psi}$$

برای یک میدان مغناطیسی، **B**، یکنواخت می توان بردار **A** را به صورت زیر در نظر گرفت:

$$\mathbf{A} = -1/2\,\mathbf{r} \times \mathbf{B}$$

دقت شود که انتخاب فوق یک انتخاب منحصر به فرد نیست، چون اگر ∇f هم به **A** اضافه شود باز هم $\nabla\times\mathbf{A}$ میدان **B** را به دست خواهد داد. اما به هر حال انتخاب فوق یک انتخاب مناسب است.

می توان درستی انتخاب فوق را به صورت زیر تحقیق کرد:

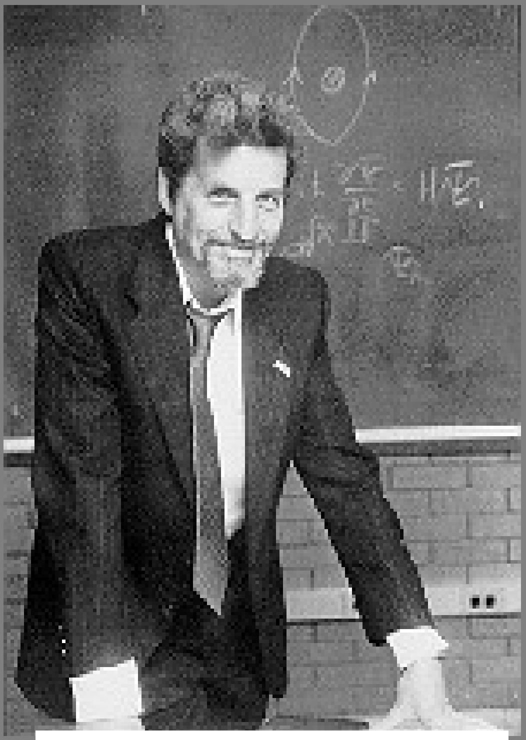
$$\mathbf{A} = -1/2\begin{pmatrix} yB_z - zB_y & zB_x - xB_z & xB_y - yB_x \end{pmatrix}$$

$$\nabla \times \mathbf{A} = \begin{pmatrix} 1/2B_x + 1/2B_x & B_y & B_z \end{pmatrix} = \mathbf{B}$$

بنابراین جمله دوم در هامیلتونی به دست

$$-\frac{ie\hbar}{\mu c}\mathbf{A}.\nabla\Psi = \frac{ie\hbar}{2\mu c}\mathbf{r}\times\mathbf{B}.\nabla\Psi = -\frac{ie\hbar}{2\mu c}\mathbf{B}.\mathbf{r}\times\nabla\Psi$$

چندین اثر کوانتوم مکانیکی جالب توجه در ارتباط با برهمکنش یک میدان مغناطیسی وجود دارد که ما اکنون به آنها برمیگردیم.



YAKIR AHARONOV

David Joseph Bohm

معادله شرودینگر

$$\frac{1}{2\mu}\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}(\mathbf{r},t)\right]^2\Psi(\mathbf{r},t)=\big(E+e\Phi(\mathbf{r})\big)\Psi(\mathbf{r},t)$$

به نظر می رسد که اصل ناوردایی پیمانه ای را نقض می کند، زیرا پتانسیل برداری در معادله شرودینگر ظاهر شده است و بنابراین هامیلتونی دستگاه که تحت تبدیل پیمانه ای

$$\mathbf{A}(\mathbf{r},t) \longrightarrow \mathbf{A}(\mathbf{r},t) + \nabla f(\mathbf{r},t)$$

به صورت زیر تبدیل می شود

$$\frac{1}{2\mu}\left[\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}(\mathbf{r},t)\right]^2 \xrightarrow{\mathbf{A}\to\mathbf{A}+\nabla f} \frac{1}{2\mu}\left[\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}(\mathbf{r},t) + \frac{e}{c}\nabla f(\mathbf{r},t)\right]^2$$

دیگر ناوردا باقی نمی ماند.

اما هنوز این امکان وجود دارد که ناوردایی پیمانه ای را حفظ کنیم.

برای حفظ ناوردایی پیمانه‌ای چنین استدلال می کنیم که یک تغییر تابع موج به اندازه یک عامل فاز نتیجه یا تغییر فیزیکی عمده ای در پاسخ نهایی ایجاد نخواهد

عامل فاز فوق حتی می تواند به **r** هم بستگی داشته باشد.

بنابراین اگر هنگام

$$\mathbf{A}(\mathbf{r},t)\longrightarrow \mathbf{A}(\mathbf{r},t)+\nabla f(\mathbf{r},t)$$

تابع موج نیز

$$\Psi(\mathbf{r},t)\longrightarrow e^{i\Lambda(\mathbf{r},t)}\Psi(\mathbf{r},t)$$

آنگاه طرف چپ معادله شرودینگر به

$$\frac{1}{2\mu}\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}+\frac{e}{c}\nabla f\right].\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}+\frac{e}{c}\nabla f\right]e^{i\Lambda}\Psi$$

$$=\frac{1}{2\mu}\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}+\frac{e}{c}\nabla f\right].\left[e^{i\Lambda}\left(\frac{\hbar}{i}\nabla\Psi+\frac{e}{c}\mathbf{A}\Psi+\frac{e}{c}\nabla f\Psi+\hbar\nabla\Lambda\Psi\right)\right]$$

$$=\frac{1}{2\mu}e^{i\Lambda}\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}+\frac{e}{c}\nabla f+\hbar\nabla\Lambda\right]^2\Psi$$

$$\Lambda(\mathbf{r},t)=-\frac{e}{\hbar c}f(\mathbf{r},t)$$

پس برای ناوردا نگاه داشتن هامیلتونی کافی است که

$$\Psi(\mathbf{r},t)\longrightarrow e^{-i\frac{e}{\hbar c}f(\mathbf{r},t)}\Psi(\mathbf{r},t)$$

انتخاب فوق معادل تبدیل زیر است:

$$\frac{1}{2\mu}e^{i\Lambda}\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}+\cancel{\frac{e}{c}\nabla f}+\cancel{\hbar\nabla\Lambda}\right]^2\Psi$$

چون در این صورت جملات سوم و چهارم موجود در طرف چپ هامیلتونی یکدیگر را حذف خواهند کرد:

$$=\frac{1}{2\mu}e^{i\Lambda}\left[\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}\right]^2\Psi$$

در یک ناحیه تهی از میدان که

$$\nabla \times \mathbf{A} = 0$$

ایجاب می کند:

پتانسیل برداری را می توان به صورت گرادیان

$$\mathbf{A} = \nabla f$$

بنابراین در یک ناحیه تهی از میدان ما حرکت الکترون را می

الف) یا این که حضور میدان مغناطیسی را کلا در نظر نگیریم و بنویسیم:

$$\left[\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla\right)^2 + V(\mathbf{r})\right]\Psi = E\Psi$$

و این معادله شرودینگر را حل کنیم

ب) با این که معادله شرودینگر را با پتانسیل برداری به صورت زیر بنویسیم

$$\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}\right)^{2}\Psi'+V(\mathbf{r})\Psi'=E\Psi'$$

و فرض کنیم

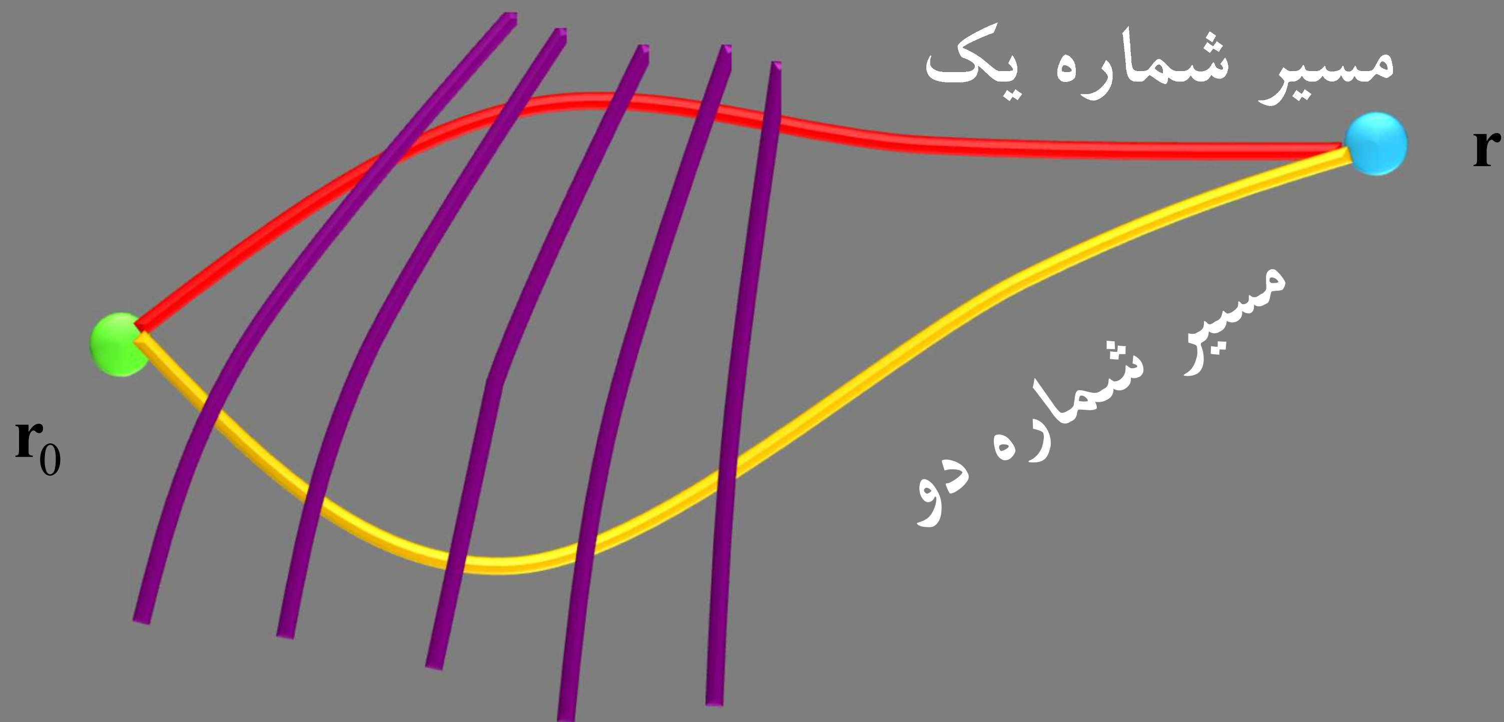
$$\Psi'=e^{-\frac{ie}{\hbar c}f}\Psi$$

تابع **f** را می توان بر حسب **A** با حل معادله زیر به صورت انتگرالی نوشت:

$$\mathbf{A}=\nabla f \Rightarrow f(\mathbf{r},t)=\int^{r} d\mathbf{r}'.\mathbf{A}(\mathbf{r}',t)$$

که در آن مسیر انتگرال گیری از یک نقطه ثابت دلخواه مانند صفر (مبداء) یا بینهایت تا نقطه **r** به طور دلخواه انتخاب شده است.

انتگرال فوق فقط اگر **B=0** باشد دارای معنی است زیرا تفاوت انتگرال در امتداد دو مسیر متفاوت 1 و 2 عبارت است از:



$$f(\mathbf{r},t)=\int_1 d\mathbf{r}'.\mathbf{A}(\mathbf{r}',t)+\int_2 d\mathbf{r}'.\mathbf{A}(\mathbf{r}',t)=\oint d\mathbf{r}'.\mathbf{A}(\mathbf{r}',t)$$

$$=\int_S d\mathbf{S}.\nabla'\times\mathbf{A}(\mathbf{r}',t)=\int_S \mathbf{B}.d\mathbf{S}=\Phi=\text{Magnetic Field Flux}$$

که در آن از قضیه استوکس استفاده شده است و ϕ فلوی مغناطیسی از سطحی است
که بر روی دو مسیر 1 و 2 بنا شده است.

بنابراین تنها اگر ϕ صفر باشد آنگاه عامل فاز **exp(-ie/(ħc))f** مستقل از مسیر خواهد شد.

چنین استقلالی از مسیر ضروری است اگر اسرار داشته باشیم

پس اگر **B=0** باشد آنگاه استقلال از مسیر وجود خواهد داشت.

اگر ϕ مخالف صفر باشد یعنی مسیرهای انتخاب شده شامل یک فلو باشد، آنگاه توابع موج الکترونهای در حال حرکت در طول دو مسیر فازهای متفاوتی خواهند داشت.

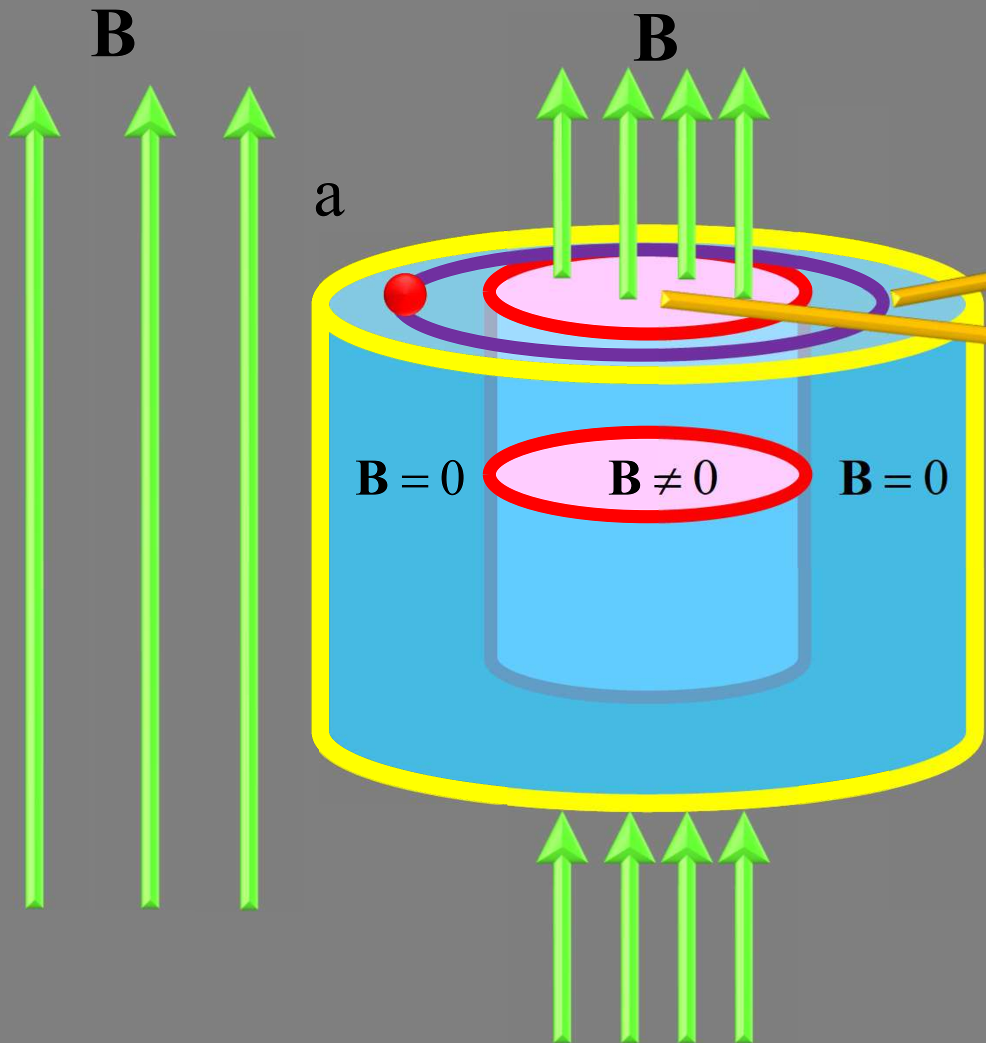
یک نتیجه جالب این است که اگر یک الکترون در یک ناحیه تهی از میدان حرکت کند (که البته نقاط ابتدائی و انتهایی مسیر خیلی راحت به هم وصل شده باشند بلکه یک حفره شامل فلوی غیر صفر را احاطه کنند) آنگاه وقتی که یک مدار بسته را طی می کند و به نقطه اول خود باز می گردید تابع موج آن به حالت اول باز نمی گردد بلکه به اندازه فاز زیر در آنها اختلاف فاز ظاهر خواهد شد:

$$\exp\left(\frac{ie\Phi}{\hbar c}\right)$$

$$\Phi = \frac{2\pi\hbar c}{e} n$$

شرط تک مقداره بودن تابع موج که هم ارز یک شدن ضریب فاز است ایجاب می کند

شده کوانتیزه باشد



مدار حرکت الکترون در

حفره

خالی از

میدان

میدان مغناطیسی که فقط اجازه حضور در درون حفره و خارج از استوانه خارجی را دارد مقدار آن

در ناحیه توپر استوانه صفر

خلاصه این بخش در مسیر حرکت الکترون:

$$\mathbf{B}=0 \;\Rightarrow\; \nabla\times\mathbf{A}=0 \;\Rightarrow\; \mathbf{A}=\nabla f$$

$$f(\mathbf{r},t)=\int_a^a d\mathbf{r}'.\mathbf{A}(\mathbf{r}',t)=\oint d\mathbf{r}'.\mathbf{A}(\mathbf{r}',t)$$

$$=\int_S d\mathbf{S}.\nabla'\times\mathbf{A}(\mathbf{r}',t)=\int_S \mathbf{B}.d\mathbf{S}=\Phi\neq 0$$

$$\Psi'=e^{-\frac{ie}{\hbar c}f}\Psi=e^{-\frac{ie}{\hbar c}\Phi}\Psi$$

B در مسیر حرکت الکترون صفر است ⬅ می توان از معادله شرودینگر در غیاب میدان مساله را حل کرد

⬅ ψ باید تک مقداره باشد ⬅ از طرفی می توان مساله را با معادله شرودینگر در حضور میدان نیز حل کرد که نتیجه آن ψ` است.

⬅ هنوز باید مساله حتی با حل در حضور میدان تک مقداره باشد زیرا واقعا **B=0** است.

پس از تک مقداره بودن

نتیجه می گیریم که:

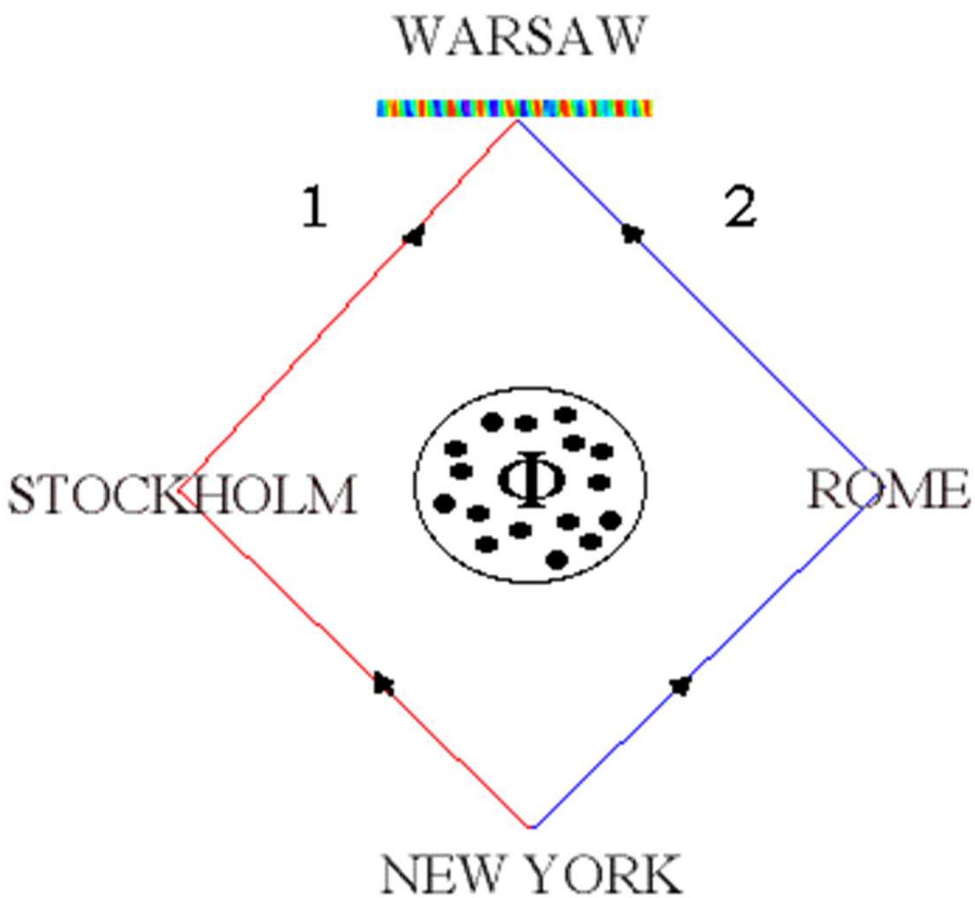
$$\Psi' = e^{-\frac{ie}{\hbar c}\Phi}\Psi$$

$$e^{-\frac{ie}{\hbar c}\Phi} = 1 \Rightarrow e^{-\frac{ie}{\hbar c}\Phi} = e^{-2n\pi i} \Rightarrow \frac{e}{\hbar c}\Phi = 2n\pi$$

$$\Rightarrow \Phi = \frac{2\pi\hbar c}{e}n$$

$$\Rightarrow \Phi_n = \frac{2\pi\hbar c}{e}n; \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

AHARONOV-BOHM EFFECT



$$\Delta\Theta = \Theta_1 - \Theta_2 = \frac{e}{\hbar}\,\Phi$$

چنین وضعیتی در حرکت الکترونها در یک حلقه ابررسانا که ناحیه ای شامل یک فلوی مغناطیسی را دربردارد به وقوع می پیوندد.

آزمایشات اولیه در سال 1961 بر اساس

یک حلقه ساخته شده از یک ابررسانا، در یک

دما در ابتدا بالاتر از دمای

دمای بحرانی دمایی است که خاصیت ابررسانایی

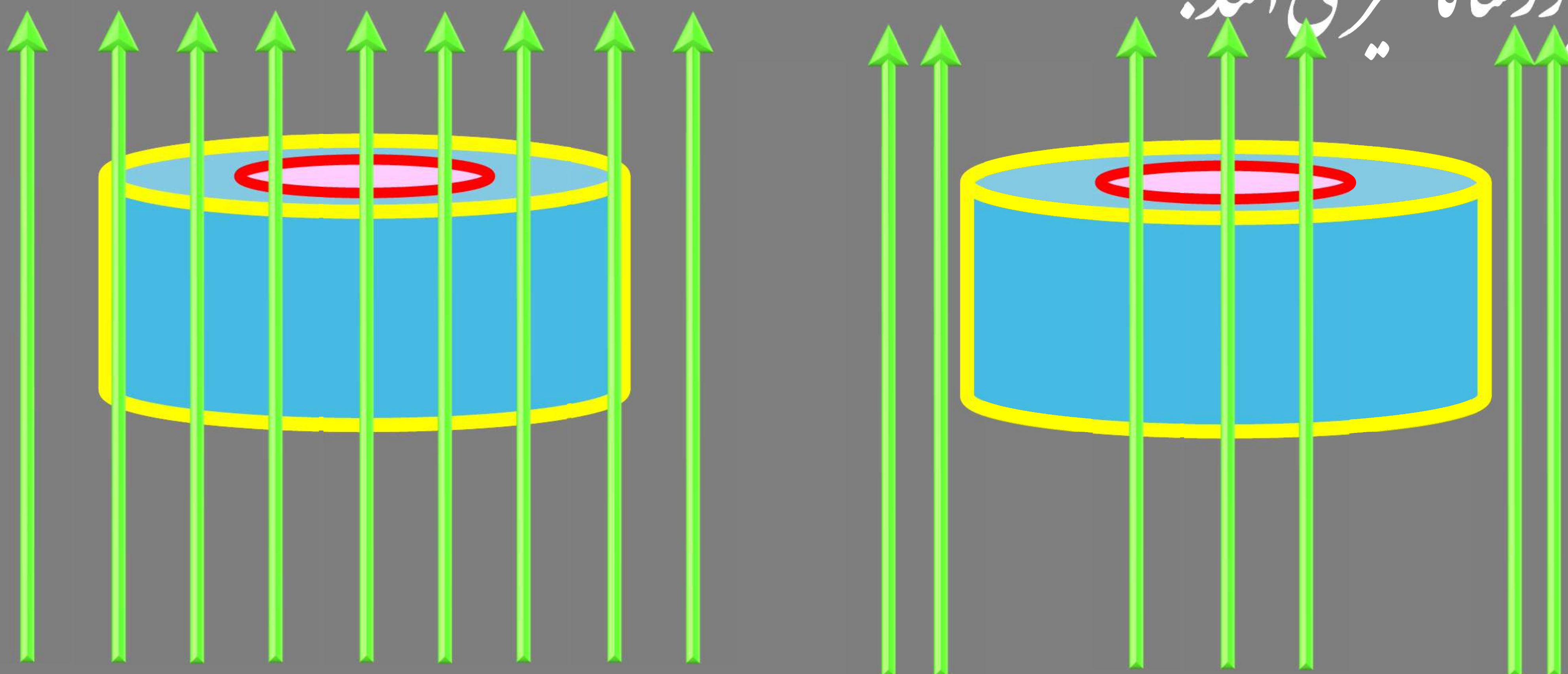
ابررساناها خطوط میدان مغناطیسی را قطع نمی کنند.

این اثر را اثر مایسنر

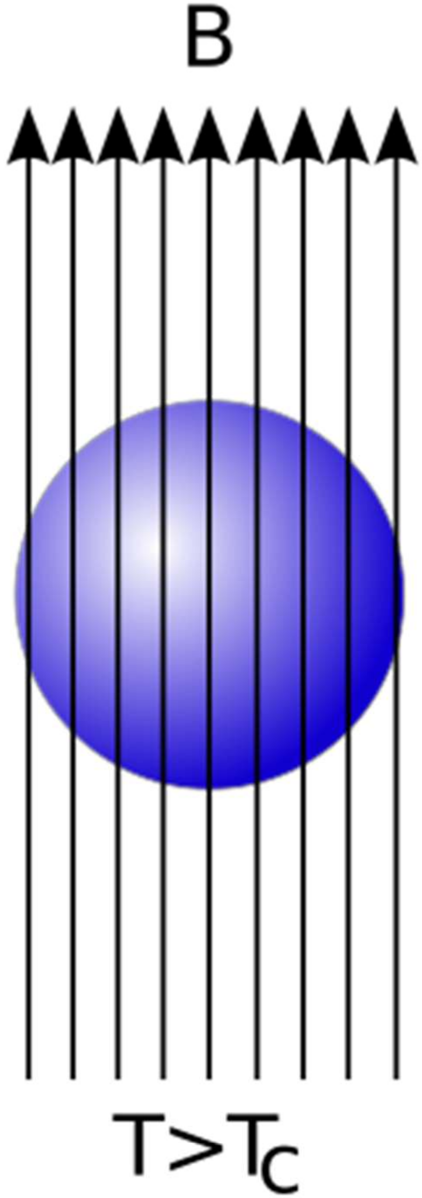
وقتی که حلقه تا زیر دمای بحرانی سرد شود، آنگاه

در این صورت میدان مغناطیسی درون حلقه

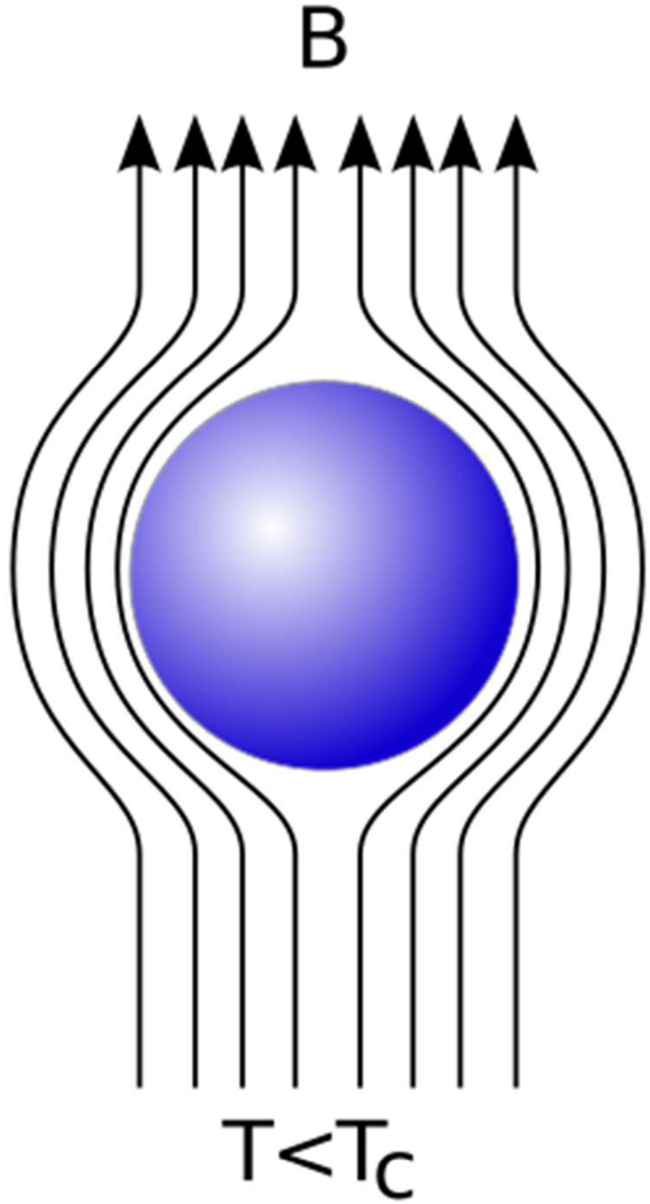
ابررسانا گیر می افتد:



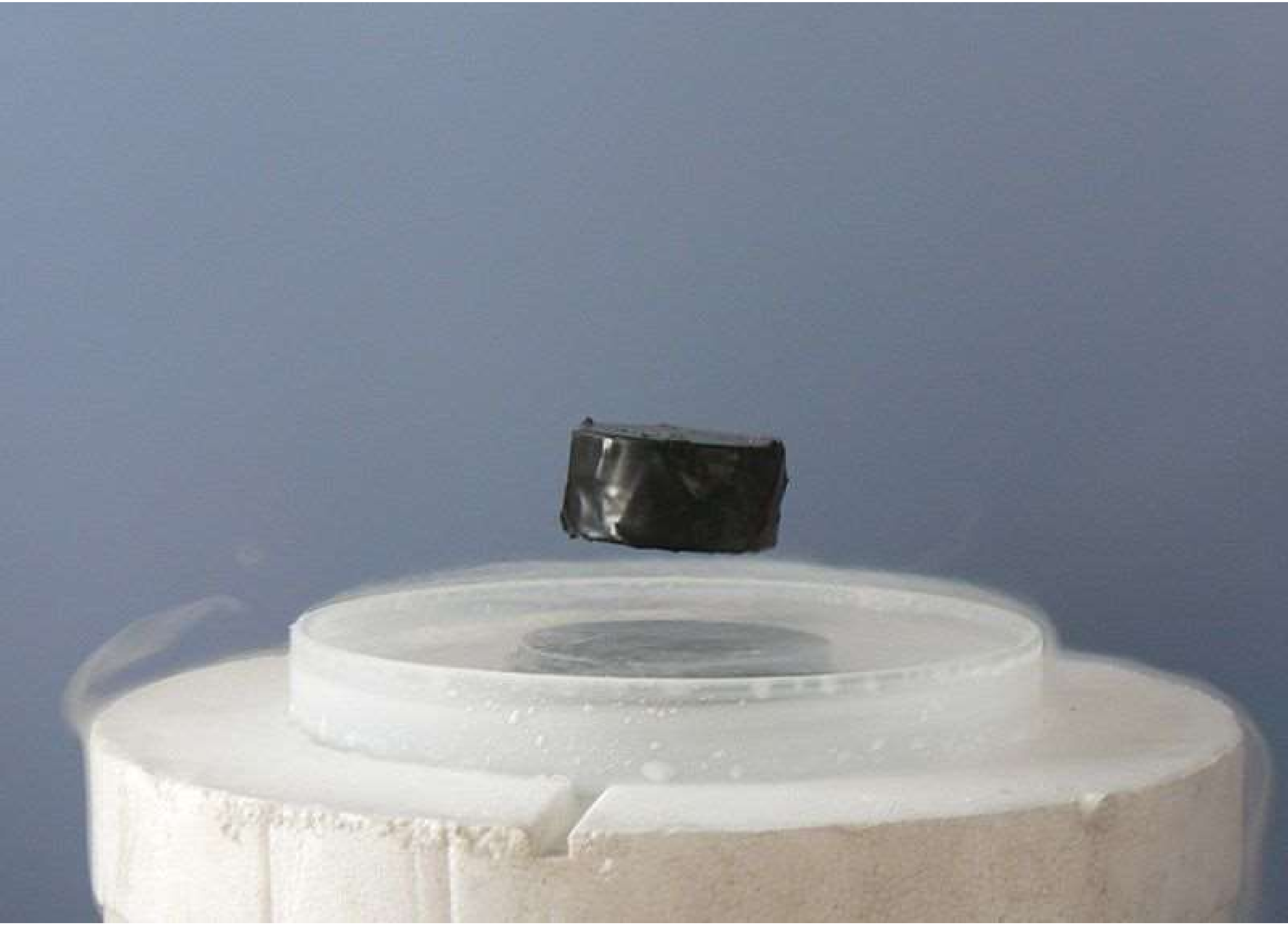
$T > T_c$ $T < T_c$



جسم در این حالت شبیه فلز رفتار می کند و میدان مغناطیسی



جسم در این حالت ابررسانا می شود و اجازه عبور میدان مغناطیسی را می



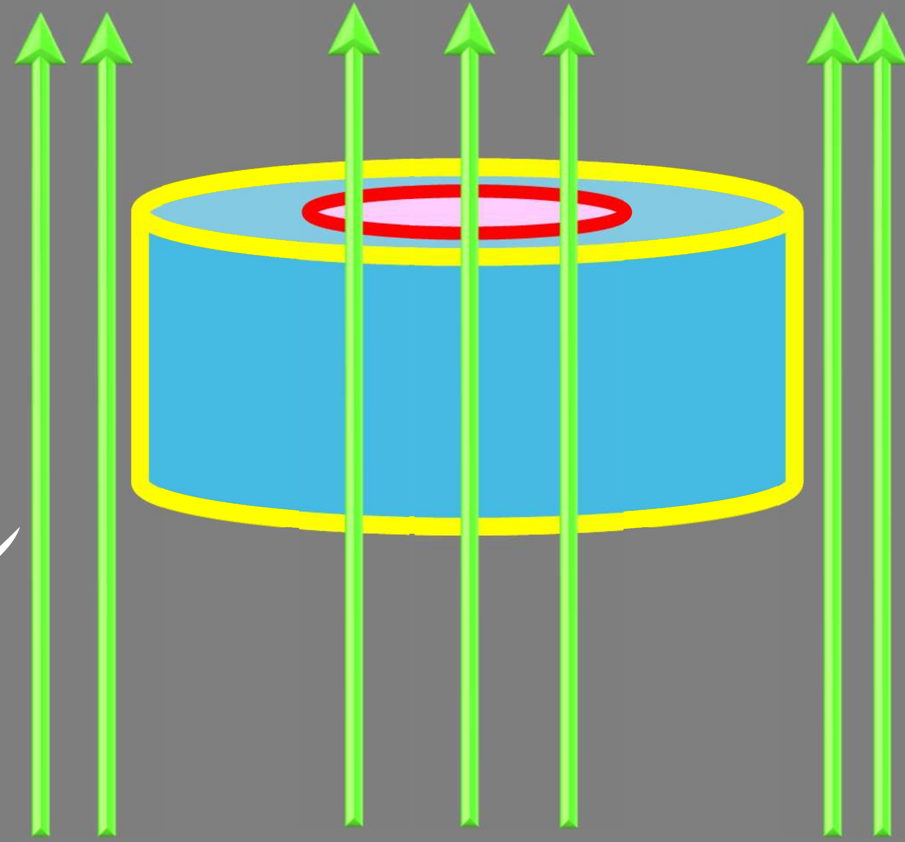
تجربه نشان می دهد که فلوی میدان مغناطیسی به صورت زیر کوانتیزه می شود:

$$\Phi_n = \frac{2\pi\hbar c}{(2e)} n$$

که با رابطه ای که ما به دست آورده ایم اندکی تفاوت

$$\Phi_n = \frac{2\pi\hbar c}{e} n$$

یعنی در تجربه به جای بار الکترون دو برابر بار الکترون ظاهر شده است.



$T < T_c$

خطوطی که در حفره گیر افتاده اند همانهایی هستند که کوانتیزه می شوند.

دو برابر شدن بار الکترون که در رابطه تجربی وجود دارد با درک فعلی ما از

درک فعلی ما می گوید که زوجهای کوپر با دو برابر بار الکترون

یعنی الکترونها در واقع دو الکترون با اسپین مخالف یکدیگر

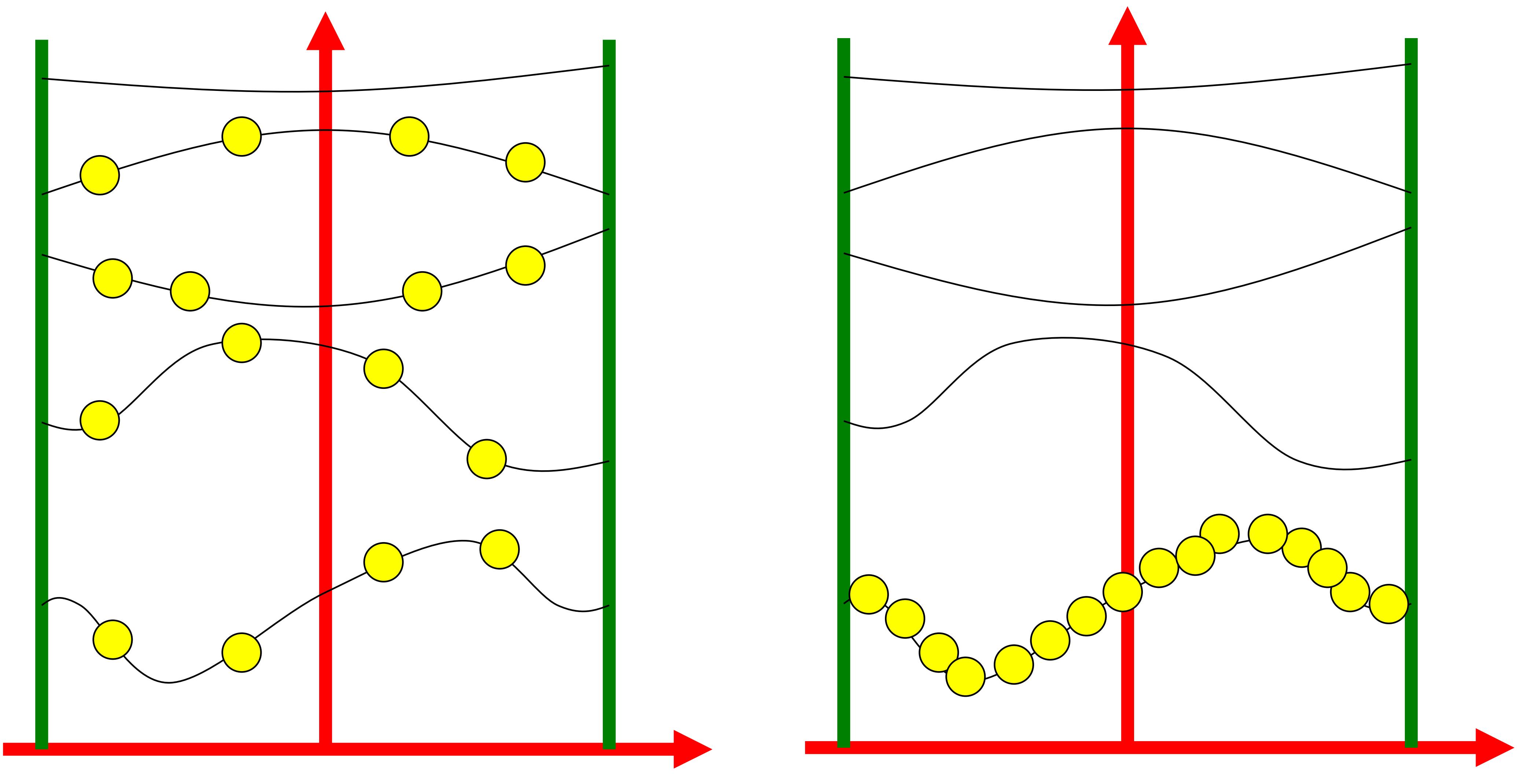
به این دو الکترون زوج

زوجهای کوپر عامل

اسپین زوجهای کوپر صفر و بنابراین مانند یک

لذا در دمای بحرانی چگالش بوز

چگالش بوز اینشتین یعنی هنگامی که می خواهیم زوجهای کوپر را ببینیم باید مانند
بوزونها با آنها رفتار کنیم و همه را در حالت زمینه در پایین ترین تراز انرژی



$T > T_c$ $T < T_c$

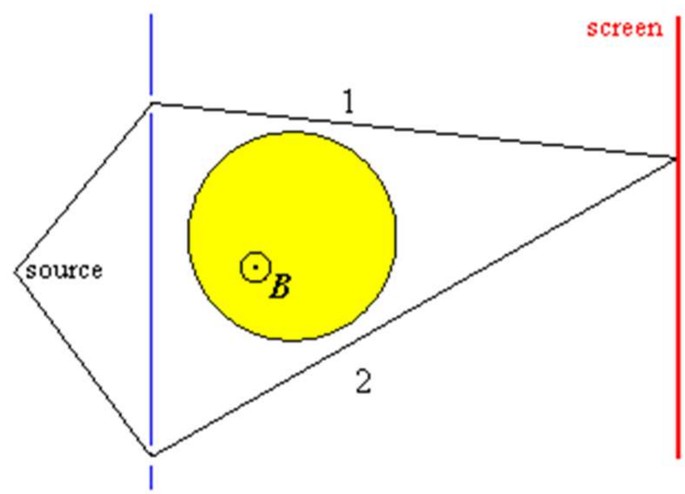
نمایش آشکار دیگری از وابستگی فاز تابع موج به فلو در اصول می تواند در یک آزمایش تداخل شبحی که در آن یک سلونوئید در برگیرنده یک میدان

طرح تداخل روی صفحه ناشی از برهمنهی دو

$$\Psi = \Psi_1 + \Psi_2$$

که در آن ψ_1 بخشی از تابع موج الکترون در مسیر **1** و ψ_2 آن بخش از تابع موج الکترون در مسیر 2 است.

$$\Psi = \Psi_1 \exp\left\{\frac{ie}{\hbar c}\int_1 d\mathbf{r}.\mathbf{A}\right\} + \Psi_2 \exp\left\{\frac{ie}{\hbar c}\int_2 d\mathbf{r}.\mathbf{A}\right\}$$



$$\Psi = \Psi_1 \exp\left\{\frac{ie}{\hbar c}\int_1 d\mathbf{r}.\mathbf{A}\right\} + \Psi_2 \exp\left\{\frac{ie}{\hbar c}\int_2 d\mathbf{r}.\mathbf{A}\right\}$$

$$\Psi = \left[\Psi_1 \exp\left\{\frac{ie}{\hbar c}\left(\int_1 d\mathbf{r}.\mathbf{A} - \int_2 d\mathbf{r}.\mathbf{A}\right)\right\} + \Psi_2\right] \exp\left\{\frac{ie}{\hbar c}\int_2 d\mathbf{r}.\mathbf{A}\right\}$$

$$\Psi = \left[\Psi_1 \exp\left\{\frac{ie}{\hbar c}\Phi\right\} + \Psi_2\right] \exp\left\{\frac{ie}{\hbar c}\int_2 d\mathbf{r}.\mathbf{A}\right\}$$

طرح تداخل در حضور میدان مغناطیسی

$$\Psi = \Psi_1 + \Psi_2$$

طرح تداخل در غیاب میدان مغناطیسی

$$|\Psi(B=0)|^2 = |\Psi_1 + \Psi_2|^2$$

$$|\Psi(B\neq 0)|^2 = \left|\Psi_1 \exp\left\{\frac{ie}{\hbar c}\Phi\right\} + \Psi_2\right|^2$$

پس طرح تداخل در حضور میدان بستگی به فلوی میدان دارد و با طرح تداخل در غیاب میدان تفاوت دارد.

نکته اخیر ابتدا به صورت نظری در مقاله زیر پیش بینی شد:

THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 115, NO. 3 AUGUST 1, 1959

Significance of Electromagnetic Potentials in the Quantum Theory

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H. H. Wills Physics Laboratory, University of Bristol, Bristol, England
(Received May 28, 1959; revised manuscript received June 16, 1959)

In this paper, we discuss some interesting properties of the electromagnetic potentials in the quantum domain. We shall show that, contrary to the conclusions of classical mechanics, there exist effects of potentials on charged particles, even in the region where all the fields (and therefore the forces on the particles) vanish. We shall then discuss possible experiments to test these conclusions; and, finally, we shall suggest further possible developments in the interpretation of the potentials.

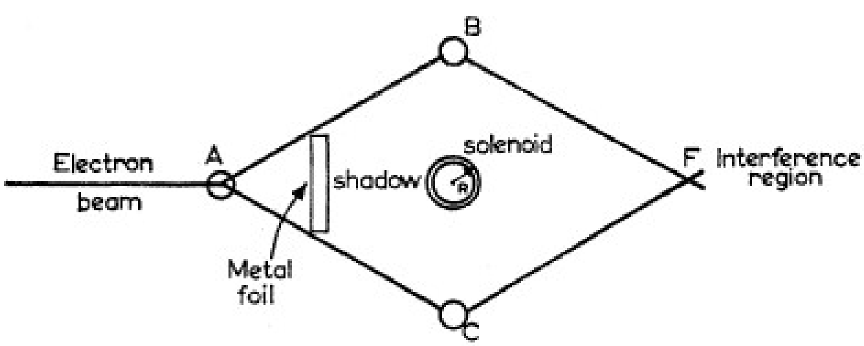


FIG. 2. Schematic experiment to demonstrate interference with time-independent vector potential.

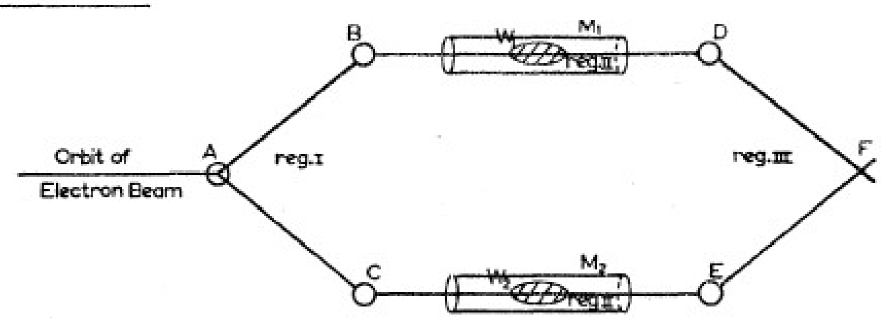


FIG. 1. Schematic experiment to demonstrate interference with time-dependent scalar potential. A, B, C, D, E: suitable devices to separate and divert beams. W_1, W_2: wave packets. M_1, M_2: cylindrical metal tubes. F: interference region.

و سپس به صورت تجربی در مقاله زیر مورد تایید قرار گرفت:

VOLUME 5, NUMBER 1 PHYSICAL REVIEW LETTERS JULY 1, 1960

SHIFT OF AN ELECTRON INTERFERENCE PATTERN BY ENCLOSED MAGNETIC FLUX

R. G. Chambers
H. H. Wills Physics Laboratory, University of Bristol, Bristol, England
(Received May 27, 1960)

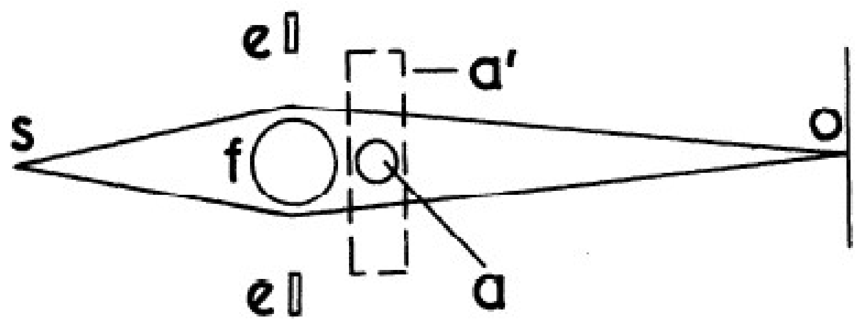


FIG. 1. Schematic diagram of interferometer, with source s, observing plane o, biprism e, f, and confined and extended field regions a and a'.

Magnetic Monopole

We conclude this section with one of the most remarkable predictions of quantum physics, which has yet to be verified experimentally.

An astute student of classical electrodynamics may be struck by the fact that there is a strong symmetry between E and B, yet a magnetic charge—commonly referred to as a magnetic monopole—analogous to electric charge is peculiarly absent in Maxwell's equations.

The source of a magnetic field observed in nature is either a moving electric charge or a static magnetic dipole, never a static magnetic charge.

Instead of

$$\nabla \cdot \mathbf{B} = 4\pi\rho_M$$

analogous to

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$\nabla.B$ actually vanishes in the usual way of writing Maxwell's equations.

Quantum mechanics does not predict that a magnetic monopole must exist.

However, it unambiguously requires that if a magnetic monopole is ever found in nature, the magnitude of magnetic charge must be quantized in terms of e, ħ, and c, as we now demonstrate.

Suppose there is a point magnetic monopole, situated at the origin, of strength e_M analogous to a point electric charge.

The static magnetic field is then given by

$$\mathbf{B} = \left(\frac{e_M}{r^2}\right)\hat{\mathbf{r}}$$

At first sight it may appear that the above magnetic field can be derived from

$$\mathbf{A} = \left[\frac{e_M(1-\cos\theta)}{r\sin\theta}\right]\hat{\boldsymbol{\phi}}$$

Recall the expression for curl in spherical coordinates:

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

But the above speculated vector potential A has one difficulty--it is singular on the negative z-axis ($\theta=\pi$).

In fact, it turns out to be impossible to construct a singularity-free potential valid everywhere for this problem.

To see this we first note "Gauss's law"

$$\int_{\text{closed surface}} \mathbf{B} \cdot d\boldsymbol{\sigma} = 4\pi e_M$$

for any surface boundary enclosing the origin at which the magnetic monopole is located.

On the other hand, if A were nonsingular, we would everywhere have

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Hence

$$\int_{\text{closed surface}} \mathbf{B}\cdot d\boldsymbol{\sigma} = \int_{\text{volume inside}} \nabla\cdot(\nabla\times\mathbf{A})\, d^3x = 0$$

in contradiction with:

$$\int_{\text{closed surface}} \mathbf{B}\cdot d\boldsymbol{\sigma} = 4\pi e_M$$

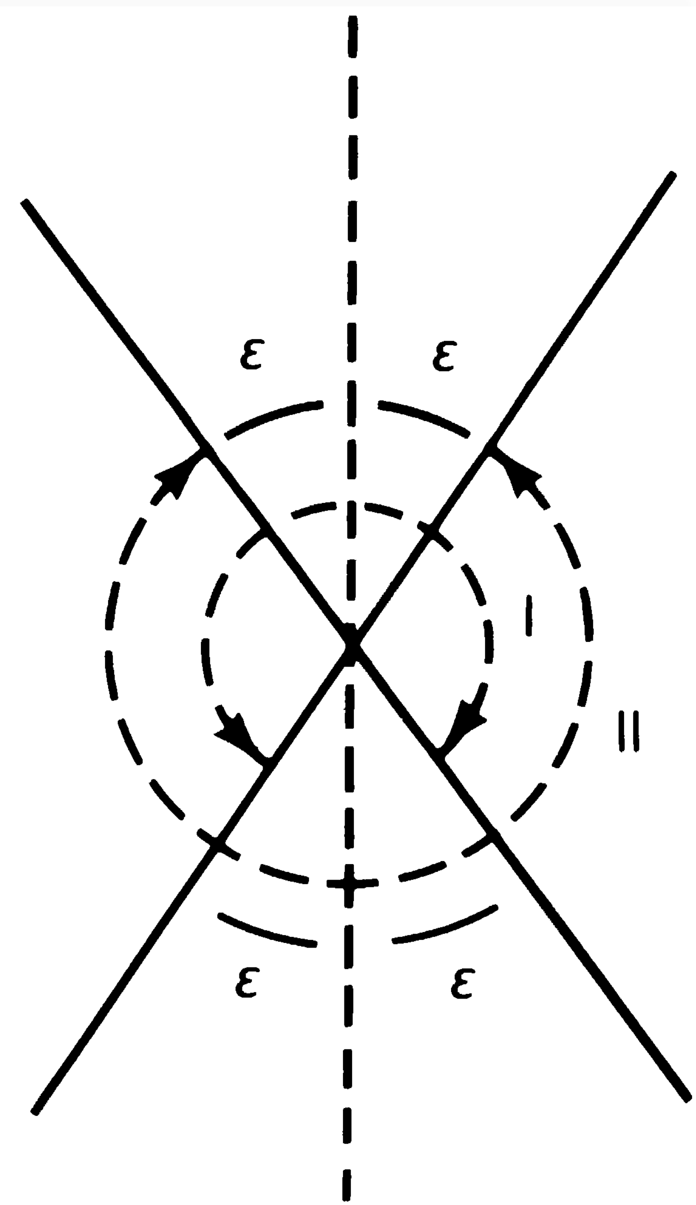
However, one might argue that because the vector potential is just a device for obtaining B, we need not insist on having a single expression for A valid everywhere.

Suppose we construct a pair of potentials,

$$\mathbf{A}^{(\mathrm{I})} = \left[\frac{e_M(1-\cos\theta)}{r\sin\theta}\right]\hat{\boldsymbol{\phi}}, \qquad (\theta < \pi - \varepsilon)$$

$$\mathbf{A}^{(\mathrm{II})} = -\left[\frac{e_M(1+\cos\theta)}{r\sin\theta}\right]\hat{\boldsymbol{\phi}}, \quad (\theta > \varepsilon)$$

such that the potential A(I) can be used everywhere except inside the cone defined by θ=π-ε around the negative z-axis; likewise, the potential A(II) can be used everywhere except inside the cone θ=ε around the positive z-axis; see the next Figure.



Together they lead to the correct expression for B everywhere.

Consider now what happens in the overlap region $\varepsilon<\theta<\pi-\varepsilon$, where we may use either A(I) or A(II).

Because the two potentials lead to the same magnetic field, they must be related to each other by a gauge transformation.

To find A appropriate for this problem we first note that

$$\mathbf{A}^{(\mathrm{II})} - \mathbf{A}^{(\mathrm{I})} = -\left(\frac{2e_M}{r\sin\theta}\right)\hat{\boldsymbol{\phi}}$$

Recalling the expression for gradient in spherical coordinates,

$$\nabla\Lambda = \hat{\mathbf{r}}\frac{\partial\Lambda}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Lambda}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\Lambda}{\partial\phi}$$

we deduce that $\Lambda = -2e_M\phi$ will do the job.

Next, we consider the wave function of an electrically charged particle of charge e subjected to magnetic field of:

$$\mathbf{B} = \left(\frac{e_M}{r^2}\right)\hat{\mathbf{r}}$$

As we emphasized earlier, the particular form of the wave function depends on the particular gauge used.

In the overlap region where we may use either A(I) or A(II), the corresponding wave functions are, according to:

$$\tilde{\psi}(\mathbf{x}', t) = \exp\left[\frac{ie\Lambda(\mathbf{x}')}{\hbar c}\right]\psi(\mathbf{x}', t)$$

related to each other by

$$\psi^{(\mathrm{II})} = \exp\left(\frac{-2iee_M\phi}{\hbar c}\right)\psi^{(\mathrm{I})}$$

Wave functions $\psi^{(\mathrm{I})}$ and $\psi^{(\mathrm{II})}$ must each be single-valued because once we choose particular gauge, the expansion of the state ket in terms of the position eigenkets must be unique.

After all, as we have repeatedly emphasized, the wave function is simply an expansion coefficient for the state ket in terms of the position eigenkets.

Let us now examine the behavior of wave function $\psi^{(II)}$ on the equator $\theta=\pi/2$ with some definite radius r, which is a constant.

When we increase the azimuthal angle φ along the equator and go around once, say from $\varphi=0$ to $\varphi=2\pi$, $\psi^{(II)}$, as well as $\psi^{(I)}$, must return to its original value because each is single-valued.

According to:

$$\psi^{(\mathrm{II})} = \exp\left(\frac{-2iee_M\phi}{\hbar c}\right)\psi^{(\mathrm{I})}$$

this is possible only if

$$\frac{2ee_M}{\hbar c} = \pm N, \quad N = 0, \pm 1, \pm 2, \ldots.$$

So we reach a very far-reaching conclusion:

The magnetic charges must be quantized in units of

$$\frac{\hbar c}{2|e|} \simeq \left(\frac{137}{2}\right)|e|$$

The smallest magnetic charge possible is $\hbar c/2|e|$, where e is the electronic charge.

It is amusing that once a magnetic monopole is assumed to exist, we can use the following relation backward,

$$\frac{2ee_M}{\hbar c} = \pm N, \quad N = 0, \pm 1, \pm 2, \ldots .$$

so to speak, to explain why the electric charges are quantized, for example, why the proton charge cannot be 0.999972 times $|e|$.

We repeat once again that quantum mechanics does not require magnetic monopoles to exist. However, it unambiguously predicts that a magnetic charge, if it is ever found in nature, must be quantized in units of $\hbar c/2|e|$. The quantization of magnetic charges in quantum mechanics was first shown in 1931 by P. A. M. Dirac. The derivation given here is due to T. T. Wu and C. N. Yang.