Theory of Angular Momentum

This chapter is concerned with a systematic treatment of angular momentum and related topics.

The importance of angular momentum in modern physics can hardly be overemphasized.

A thorough understanding of angular momentum is essential in molecular, atomic, and nuclear spectroscopy; angular-momentum considerations play an important role in scattering and collision problems as well as in bound-state problems.

Furthermore, angular-momentum concepts have important generalizations—isospin in nuclear physics, SU(3), SU(2)⊗U(1) in particle physics, and so forth.

ROTATIONS AND ANGULAR MOMENTUM COMMUTATION RELATIONS

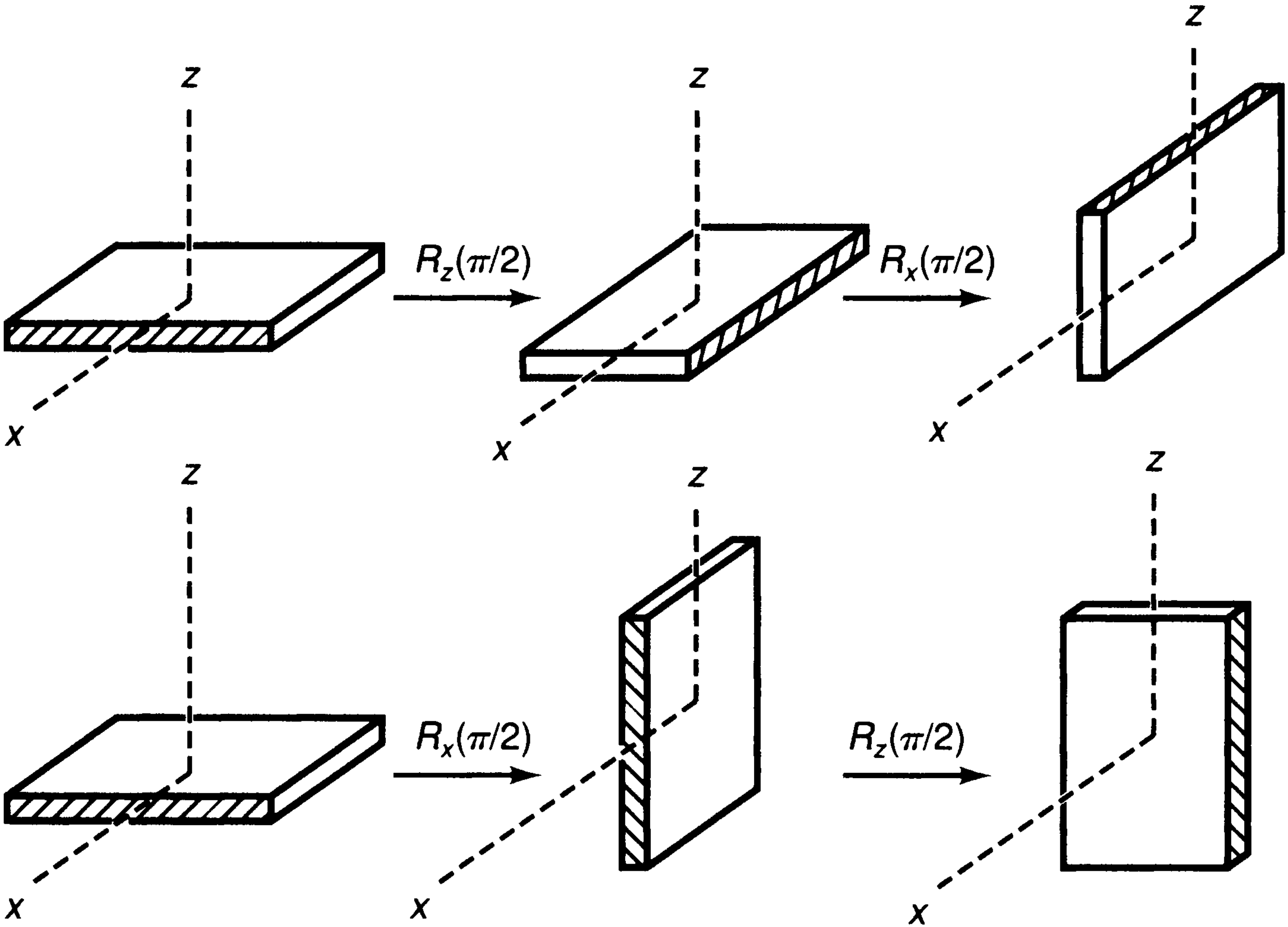
Finite Versus Infinitesimal Rotations

We recall from elementary physics that rotations about the same axis commute, whereas rotations about different axes do not.

For instance, a 30° rotation about the z-axis followed by a 60° rotation about the same z-axis is obviously equivalent to a 60° rotation followed by a 30° rotation, both about the same axis.

However, let us consider a 90° rotation about the z-axis, denoted by $R_z(\frac{\pi}{2})$, followed by a 90° rotation about the x-axis, denoted by $R_x(\frac{\pi}{2})$; compare this with a 90° rotation about the x-axis followed by a 90° rotation about the z-axis.

The net results are different, as we can see from Figure 3.1.


z
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$R_z(\pi/2)$
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Our first basic task is to work out quantitatively the manner in which rotations about different axes fail to commute.

To this end, we first recall how to represent rotations in three dimensions by 3×3 real, orthogonal matrices.

Consider a vector **V** with components V_x, V_y, and V_z.

When we rotate, the three components become some other set of numbers, V'_x, V'_y, and V'_z.

The old and new components are related via a 3×3 orthogonal matrix R:

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} & R & \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$RR^T = R^T R = 1$$

where the superscript T stands for a transpose of a matrix.

$$\sqrt{V_x^2 + V_y^2 + V_z^2} = \sqrt{V_x'^2 + V_y'^2 + V_z'^2}$$

It is a property of orthogonal matrices that is automatically satisfied.

To be definite, we consider a rotation about the z-axis by angle ϕ.

The convention we follow throughout this book is that a rotation operation affects a physical system itself, as in Figure 3.1, while the coordinate axes remain unchanged.

The angle ϕ is taken to be positive when the rotation in question is counterclockwise in the xy-plane, as viewed from the positive z-side.

If we associate a right-handed screw with such a rotation, a positive ϕ rotation around the z-axis means that the screw is advancing in the positive z-direction. With this convention, we easily verify that

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Had we adopted a different convention, in which a physical system remained fixed but the coordinate axes rotated, this same matrix with a positive ϕ would have represented a clockwise rotation of the x- and y-axes, when viewed from the positive z-side. It is obviously important not to mix the two conventions! Some authors distinguish the two approaches by using **"active rotations"** for physical systems rotated and **"passive rotations"** for the coordinate axes rotated.

We are particularly interested in an infinitesimal form of Rz:

$$R_z(\varepsilon) = \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & -\varepsilon & 0 \\ \varepsilon & 1-\frac{\varepsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where terms of order ε^3 and higher are ignored.

Likewise, we have

$$R_x(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\varepsilon^2}{2} & -\varepsilon \\ 0 & \varepsilon & 1-\frac{\varepsilon^2}{2} \end{pmatrix}$$

$$R_y(\varepsilon) = \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1-\frac{\varepsilon^2}{2} \end{pmatrix}$$

Compare now the effect of a y-axis rotation followed by an x-axis rotation with that of an x-axis rotation followed by a x-axis rotation. Elementary matrix manipulations lead to

$$R_x(\varepsilon)R_y(\varepsilon) = \begin{pmatrix} 1-\dfrac{\varepsilon^2}{2} & 0 & \varepsilon \\ \varepsilon^2 & 1-\dfrac{\varepsilon^2}{2} & -\varepsilon \\ -\varepsilon & \varepsilon & 1-\varepsilon^2 \end{pmatrix}$$

$$R_y(\varepsilon)R_x(\varepsilon) = \begin{pmatrix} 1-\dfrac{\varepsilon^2}{2} & \varepsilon^2 & \varepsilon \\ 0 & 1-\dfrac{\varepsilon^2}{2} & -\varepsilon \\ -\varepsilon & \varepsilon & 1-\varepsilon^2 \end{pmatrix}$$

From these two last equations we have the first important result:

Infinitesimal rotations about different axes do commute if terms of order ε^2 and higher are ignored.

The second and even more important result concerns the manner in which rotations about different axes fail to commute when terms of order ε^2 are kept:

$$R_x(\varepsilon)R_y(\varepsilon)-R_y(\varepsilon)R_x(\varepsilon)=\begin{pmatrix}0 & -\varepsilon^2 & 0\\ \varepsilon^2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$=R_z(\varepsilon^2)-1,$$

where all terms of order higher than ε^2 have been ignored throughout this derivation.

We also have

$$1 = R_{\text{any}}(0)$$

where any stands for any rotation axis.

Thus the final result can be written as

$$R_x(\varepsilon)R_y(\varepsilon) - R_y(\varepsilon)R_x(\varepsilon) = R_z(\varepsilon^2) - R_{\text{any}}(0)$$

This is an example of the commutation relations between rotation operations about different axes, which we will use later in deducing the angular momentum commutation relations in quantum mechanics.

Infinitesimal Rotations in Quantum Mechanics

So far we have not used quantum-mechanical concepts.

The matrix R is just a 3 × 3 orthogonal matrix acting on a vector V written in column matrix form.

We must now understand how to characterize rotations in quantum mechanics.

Because rotations affect physical systems, the state ket corresponding to a rotated system is expected to look different from the state ket corresponding to the original unrotated system.

Given a rotation operation R, characterized by a 3 × 3 orthogonal matrix R, we associate an operator $\mathscr{D}$(R) in the appropriate ket space such that

$$|\alpha\rangle_R = \mathscr{D}(R)|\alpha\rangle$$

where $|\alpha\rangle_R$ and $|\alpha\rangle$ stand for the kets of the rotated and original system, respectively.

Note that the 3×3 orthogonal matrix R acts on a column matrix made up of the three components of a classical vector, while the operator $\mathscr{D}$(R) acts on state vectors in ket space.

The matrix representation of $\mathscr{D}$(R), which we will study in great detail in the subsequent sections, depends on the dimensionality N of the particular ket space in question.

For N = 2, appropriate for describing a spin ½ system with no other degrees of freedom, $\mathscr{D}$(R) is represented by a 2 × 2 matrix; for a spin 1 system, the appropriate representation is a 3 × 3 unitary matrix, and so on.

To construct the rotation operator $\mathscr{D}$(R), it is again fruitful to examine first its properties under an infinitesimal rotation.

We can almost guess how we must proceed by analogy.

In both translations and time evolution, which we studied in Sections 1.6 and 2.1, respectively, the appropriate infinitesimal operators could be written as

$$U_{\varepsilon} = 1 - iG\varepsilon$$

with a Hermitian operator G. Specifically,

$$G \rightarrow \frac{p_x}{\hbar}, \qquad \varepsilon \rightarrow dx'$$

for an infinitesimal translation by a displacement dx' in the x-direction, and

$$G \rightarrow \frac{H}{\hbar}, \qquad \varepsilon \rightarrow dt$$

for an infinitesimal time evolution with time displacement dt.

We know from classical mechanics that angular momentum is the generator of rotation in much the same way as momentum and Hamiltonian are the generators of translation and time evolution, respectively.

We therefore define the angular momentum operator J_k in such a way that the operator for an infinitesimal rotation around the k th axis by angle $d\phi$ can be obtained by letting

$$G \to \frac{J_k}{\hbar}, \qquad \varepsilon \to d\phi$$

in

$$U_\varepsilon = 1 - iG\varepsilon$$

With J_k taken to be Hermitian, the infinitesimal rotation operator is guaranteed to be unitary and reduces to the identity operator in the limit $d\phi \rightarrow 0$.

More generally, we have

$$\mathscr{D}(\hat{\mathbf{n}}, d\phi) = 1 - i\left(\frac{\mathbf{J}\cdot\hat{\mathbf{n}}}{\hbar}\right) d\phi$$

for a rotation about the direction characterized by a unit vector $\hat{n}$ by an infinitesimal angle $d\phi$.

We stress that in this book we do not define the angular-momentum operator to be $\mathbf{x}\times\mathbf{p}$.

This is important because spin angular momentum, to which our general formalism also applies, has nothing to do with x_i and p_j.

Put in another way, in classical mechanics one can prove that the angular momentum defined to be $\mathbf{x}\times\mathbf{p}$ is the generator of a rotation; in contrast, in quantum mechanics we define J so that the operator for an infinitesimal rotation takes the following form

$$\mathscr{D}(\hat{\mathbf{n}}, d\phi) = 1 - i\left(\frac{\mathbf{J}\cdot\hat{\mathbf{n}}}{\hbar}\right) d\phi$$

A finite rotation can be obtained by compounding successively infinitesimal rotations about the same axis.

For instance, if we are interested in a finite rotation about the z-axis by angle ϕ, we consider

$$\begin{aligned}
\mathscr{D}_z(\phi) &= \lim_{N \to \infty} \left[1 - i\left(\frac{J_z}{\hbar}\right)\left(\frac{\phi}{N}\right)\right]^N \\
&= \exp\left(\frac{-iJ_z\phi}{\hbar}\right) \\
&= 1 - \frac{iJ_z\phi}{\hbar} - \frac{J_z^2\phi^2}{2\hbar^2} + \cdots .
\end{aligned}$$

In order to obtain the angular-momentum commutation relations, we need one more concept.

As we remarked earlier, for every rotation R represented by a 3 × 3 orthogonal matrix R there exists a rotation operator $\mathscr{D}$(R) in the appropriate ket space.

We further postulate that $\mathscr{D}$(R) has the same group properties as R:

Identity: $R \cdot 1 = R \Rightarrow \mathscr{D}(R) \cdot 1 = \mathscr{D}(R)$

Closure: $R_1 R_2 = R_3 \Rightarrow \mathscr{D}(R_1)\mathscr{D}(R_2) = \mathscr{D}(R_3)$

Inverses: $RR^{-1} = 1 \Rightarrow \mathscr{D}(R)\mathscr{D}^{-1}(R) = 1$

$R^{-1}R = 1 \Rightarrow \mathscr{D}^{-1}(R)\mathscr{D}(R) = 1$

Associativity: $R_1(R_2R_3) = (R_1R_2)R_3 = R_1R_2R_3$

$$\Rightarrow \mathscr{D}(R_1)[\mathscr{D}(R_2)\mathscr{D}(R_3)]$$

$$= [\mathscr{D}(R_1)\mathscr{D}(R_2)]\mathscr{D}(R_3)$$

$$= \mathscr{D}(R_1)\mathscr{D}(R_2)\mathscr{D}(R_3).$$

Let us now return to the fundamental commutation relations for rotation operations written in terms of the R matrices.

Its rotation operator analogue would read

$$\left(1-\frac{iJ_x\varepsilon}{\hbar}-\frac{J_x^2\varepsilon^2}{2\hbar^2}\right)\left(1-\frac{iJ_y\varepsilon}{\hbar}-\frac{J_y^2\varepsilon^2}{2\hbar^2}\right)$$
$$-\left(1-\frac{iJ_y\varepsilon}{\hbar}-\frac{J_y^2\varepsilon^2}{2\hbar^2}\right)\left(1-\frac{iJ_x\varepsilon}{\hbar}-\frac{J_x^2\varepsilon^2}{2\hbar^2}\right)=1-\frac{iJ_z\varepsilon^2}{\hbar}-1$$

Terms of order ε automatically drop out.

Repeating this kind of argument with rotations about other axes, we obtain

$$[J_x, J_y] = i\hbar J_z$$

Repeating this kind of argument with rotations about other axes, we obtain

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

known as the **fundamental commutation relations of angular momentum**.

In general, when the generators of infinitesimal transformations do not commute, the corresponding group of operations is said to be non-Abelian.

Because of the fundamental commutation relation, the rotation group in three dimensions is non-Abelian.

In contrast, the translation group in three dimensions is Abelian because p_i and p_j commute even with $i \neq j$.

We emphasize that in obtaining the commutation relations we have used the following two concepts:

1. J_k is the generator of rotation about the k th axis.

2. Rotations about different axes fail to commute.

It is no exaggeration to say that commutation relations summarize in a compact manner all the basic properties of rotations in three dimensions.