SPIN 1/2 SYSTEMS AND FINITE ROTATIONS

Rotation Operator for Spin ½

The lowest number, N, of dimensions in which the angular-momentum commutation relations $[J_i, J_j]$ are realized, is N =2.

You have already checked in Problem 8 of Chapter 1 that the operators defined by

$$S_x = \left(\frac{\hbar}{2}\right)\{(|+\rangle\langle -|) + (|-\rangle\langle +|)\},$$

$$S_y = \left(\frac{i\hbar}{2}\right)\{-(|+\rangle\langle -|) + (|-\rangle\langle +|)\},$$

$$S_z = \left(\frac{\hbar}{2}\right)\{(|+\rangle\langle +|) - (|-\rangle\langle -|)\}$$

satisfy the commutation relations $[J_i, J_j]$ with J_k replaced by S_k.

It is not a priori obvious that nature takes advantage of the lowest dimensional realization of $[J_i, J_j]$, but numerous experiments—from atomic spectroscopy to nuclear magnetic resonance—suffice to convince us that this is in fact the case.

Consider a rotation by a finite angle ϕ about the z-axis. If the ket of a spin ½ system before rotation is given by $|\alpha\rangle$, the ket after rotation is given by

$$|\alpha\rangle_R = \mathscr{D}_z(\phi)|\alpha\rangle$$

$$\mathscr{D}_z(\phi) = \exp\left(\frac{-iS_z\phi}{\hbar}\right).$$

To see that this operator really rotates the physical system, let us look at its effect on <Sx>. Under rotation this expectation value changes as follows:

$$\langle S_x \rangle \to {}_R\langle \alpha | S_x | \alpha \rangle_R = \langle \alpha | \mathscr{D}_z^\dagger(\phi) S_x \mathscr{D}_z(\phi) | \alpha \rangle$$

We must therefore compute

$$\exp\left(\frac{iS_z\phi}{\hbar} \right) S_x \exp\left(\frac{-iS_z\phi}{\hbar} \right)$$

For pedagogical reasons we evaluate this in two different ways.

Derivation 1: Here we use the specific form of Sx given by:

$$S_x = \left(\frac{\hbar}{2}\right)\{(|+\rangle\langle -|) + (|-\rangle\langle +|)\},$$

$$S_y = \left(\frac{i\hbar}{2}\right)\{-(|+\rangle\langle -|) + (|-\rangle\langle +|)\},$$

$$S_z = \left(\frac{\hbar}{2}\right)\{(|+\rangle\langle +|) - (|-\rangle\langle -|)\}$$

We then obtain for:

$$\exp\left(\frac{iS_z\phi}{\hbar}\right) S_x \exp\left(\frac{-iS_z\phi}{\hbar}\right)$$

$$\left(\frac{\hbar}{2}\right)\exp\left(\frac{iS_z\phi}{\hbar}\right)\{(|+\rangle\langle -|) + (|-\rangle\langle +|)\}\exp\left(\frac{-iS_z\phi}{\hbar}\right)$$

$$= \left(\frac{\hbar}{2}\right)\left(e^{i\phi/2}|+\rangle\langle -|e^{i\phi/2} + e^{-i\phi/2}|-\rangle\langle +|e^{-i\phi/2}\right)$$

$$= \frac{\hbar}{2}\left[\{(|+\rangle\langle -|) + (|-\rangle\langle +|)\}\cos\phi + i\{(|+\rangle\langle -|) - (|-\rangle\langle +|)\}\sin\phi\right]$$

$$= S_x\cos\phi - S_y\sin\phi$$

Derivation 2: Alternatively we may use the following formula:

$$\exp(iG\lambda)A\exp(-iG\lambda) = A + i\lambda[G,A] + \left(\frac{i^2\lambda^2}{2!}\right)[G,[G,A]] + \cdots + \left(\frac{i^n\lambda^n}{n!}\right)[G,[G,[G,\ldots[G,A]]]\ldots] + \cdots,$$

To evaluate:

$$\exp\left(\frac{iS_z\phi}{\hbar}\right)S_x\exp\left(\frac{-iS_z\phi}{\hbar}\right) = S_x + \left(\frac{i\phi}{\hbar}\right)\underbrace{[S_z,S_x]}_{i\hbar S_y}$$

$$+\left(\frac{1}{2!}\right)\left(\frac{i\phi}{\hbar}\right)^2\underbrace{[S_z,\underbrace{[S_z,S_x]]}_{i\hbar S_y}}_{\hbar^2 S_x} + \left(\frac{1}{3!}\right)\left(\frac{i\phi}{\hbar}\right)^3\underbrace{[S_z,\underbrace{[S_z,[S_z,S_x]]]}_{\hbar^2 S_x}}_{i\hbar^3 S_y} + \cdots$$

$$= S_x\left[1-\frac{\phi^2}{2!}+\cdots\right] - S_y\left[\phi-\frac{\phi^3}{3!}+\cdots\right] = S_x\cos\phi - S_y\sin\phi$$

Notice that in derivation 2 we used only the commutation relations for Si, so this method can be generalized to rotations of systems with angular momentum higher than ½.

For spin ½, both methods give

$$\langle S_x \rangle \to {}_R\langle \alpha | S_x | \alpha \rangle_R = \langle S_x \rangle \cos\phi - \langle S_y \rangle \sin\phi$$

where the expectation value without subscripts is understood to be taken with respect to the (old) unrotated system.

$$|\alpha\rangle_R = \mathscr{D}(R)|\alpha\rangle$$

Similarly,

$$\langle S_y \rangle \rightarrow \langle S_y \rangle \cos\phi + \langle S_x \rangle \sin\phi$$

As for the expectation value of S_z, there is no change because S_z commutes with $\mathscr{D}_z(\phi)$:

$$\langle S_z \rangle \rightarrow \langle S_z \rangle$$

Relations $<S_x>$, $<S_y>$, and $<S_z>$ are quite reasonable.

They show that rotation operator $\mathscr{D}_z(\phi)$, when applied to the state ket, does rotate the expectation value of **S** around the z-axis by angle ϕ.

In other words, the expectation value of the spin operator behaves as though it were a classical vector under rotation:

$$\langle S_k \rangle \rightarrow \sum_l R_{kl} \langle S_l \rangle$$

where R_{kl} are the elements of the 3×3 orthogonal matrix R that specifies the rotation in question.

It should be clear from our derivation 2 that this property is not restricted to the spin operator of spin ½ systems.

In general, we have

$$\langle J_k \rangle \rightarrow \sum_l R_{kl} \langle J_l \rangle$$

where J_k are the generators of rotations satisfying the angular-momentum commutation relations.

Later we will show that relations of this kind can be further generalized to any vector operator.

So far everything has been as expected.

But now, be prepared for a surprise!

We examine the effect of rotation operator $\mathscr{D}_z(\phi)$ on a general ket,

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle$$

a little more closely.

We see that

$$\exp\left(\frac{-iS_z\phi}{\hbar}\right)|\alpha\rangle = e^{-i\phi/2}|+\rangle\langle+|\alpha\rangle + e^{i\phi/2}|-\rangle\langle-|\alpha\rangle$$

The appearance of the half-angle $\phi/2$ here has an extremely interesting consequence.

Let us consider a rotation by 2π.

We then have

$$|\alpha\rangle_{R_z(2\pi)} \rightarrow -|\alpha\rangle$$

So the ket for the 360° rotated state differs from the original ket by a minus sign.

We would need a 720° ($\phi = 4\pi$) rotation to get back to the same ket with a *plus sign*.

Notice that this minus sign disappears for the expectation value of **S** because **S** is sandwiched by $|\alpha>$ and $< \alpha|$, both of which change sign.

Will this minus sign ever be observable?

Spin Precession Revisited

We now treat the problem of spin precession, already discussed in Section 2.1, from a new point of view.

We recall that the basic Hamiltonian of the problem is given by

$$H = -\left(\frac{e}{m_e c}\right)\mathbf{S}\cdot\mathbf{B} = \omega S_z$$

$$\omega \equiv \frac{|e|B}{m_e c}$$

The time-evolution operator based on this Hamiltonian is given by

$$\mathscr{U}(t,0) = \exp\left(\frac{-iHt}{\hbar}\right) = \exp\left(\frac{-iS_z\omega t}{\hbar}\right)$$

Comparing this equation with $\mathscr{D}_z(\phi)$, we see that the time-evolution operator here is precisely the same as the rotation operator in $\mathscr{D}_z(\phi)$ with ϕ set equal to ωt.

Paraphrasing $<S_x>$, $<S_y>$, and $<S_z>$, we obtain

$$\langle S_x \rangle_t = \langle S_x \rangle_{t=0} \cos \omega t - \langle S_y \rangle_{t=0} \sin \omega t,$$

$$\langle S_y \rangle_t = \langle S_y \rangle_{t=0} \cos \omega t + \langle S_x \rangle_{t=0} \sin \omega t,$$

$$\langle S_z \rangle_t = \langle S_z \rangle_{t=0}.$$

After $t = 2\pi/\omega$, the spin returns to its original direction.

This set of equations can be used to discuss the spin precession of a **muon**, an electronlike particle which, however, is 210 times as heavy. The muon magnetic moment can be determined from other experiments—for example, the hyperfine splitting in muonium, a bound state of a positive muon and an electron—to be $e\hbar/2m_\mu c$, just as expected from Dirac's relativistic theory of spin $\frac{1}{2}$ particles. (We will here neglect very small corrections that arise from quantum field theory effects). Knowing the magnetic moment we can predict the angular frequency of precession. So (3.2.19) can be and, in fact, has been checked experimentally. In practice, as the external magnetic field causes spin precession, the spin direction is analyzed by taking advantage of the fact that electrons from muon decay tend to be emitted preferentially in the direction opposite to the muon spin.

Let us now look at the time evolution of the state ket itself.

Assuming that the initial (t = 0) ket is given by

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle$$

we obtain after time

$$|\alpha, t_0 = 0; t\rangle = e^{-i\omega t/2}|+\rangle\langle+|\alpha\rangle + e^{+i\omega t/2}|-\rangle\langle-|\alpha\rangle$$

Expression $\mathscr{D}_z(\phi)$ acquires a minus sign at $t=2\pi/\omega$, and we must wait until $t = 4\pi/\omega$ to get back to the original state ket with the same sign.

To sum up, the period for the state ket is twice as long as the period for spin precession

$$\tau_{\text{precession}} = \frac{2\pi}{\omega}$$

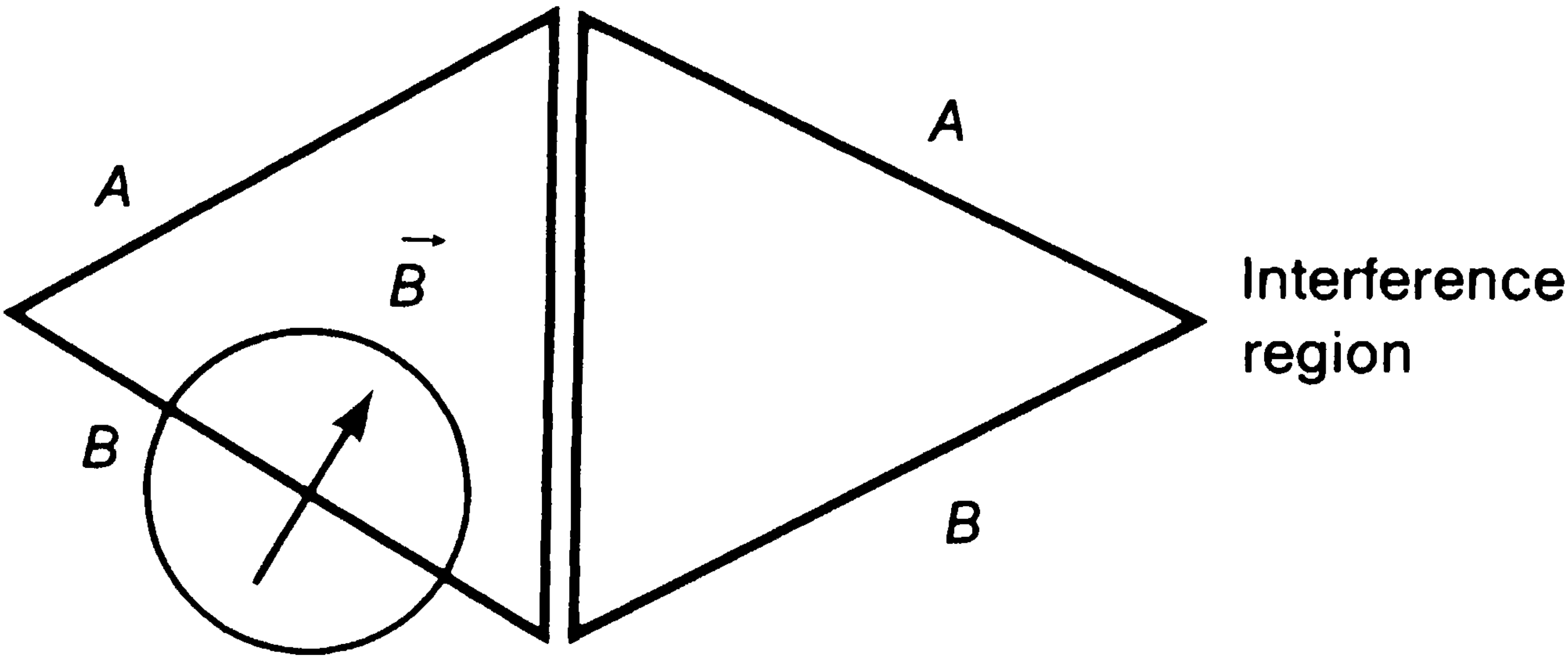
$$\tau_{\text{state ket}} = \frac{4\pi}{\omega}$$

Neutron Interferometry Experiment to Study 2π Rotations

We now describe an experiment performed to detect the minus sign in (3.2.15). Quite clearly, if every state ket in the universe is multiplied by a minus sign, there will be no observable effect. The only way to detect the predicted minus sign is to make a comparison between an unrotated state and a rotated state. As in gravity-induced quantum interference, discussed in Section 2.6, we rely on the art of neutron interferometry to verify this extraordinary prediction of quantum mechanics.

A nearly monoenergetic beam of thermal neutrons is split into two parts—path A and path B; see Figure 3.2.

Path A always goes through a magnetic-field-free region; in contrast, path B enters a small region where a static magnetic field is present.



As a result, the neutron state ket going via path B suffers a phase change $e^{-i\omega T/2}$, where T is the time spent in the $B \neq 0$ region and ω is the spin-precession frequency

$$\omega = \frac{g_n eB}{m_p c}, \quad (g_n \simeq -1.91)$$

for the neutron with a magnetic moment of $g_n e\hbar/2m_p c$, as we can see if we compare this with $\omega \equiv |e|B/m_e c$, which is appropriate for the electron with magnetic moment $e\hbar/2m_e c$.

When path A and path B meet again in the interference region of the Figure, the amplitude of the neutron arriving via path B is

$$c_2 = c_2(B=0)\,e^{\mp \imath\omega T/2}$$

while the amplitude of the neutron arriving via path A is c_1 independent of B.

So the intensity observable in the interference region must exhibit a sinusoidal variation

$$\cos\left(\frac{\mp\omega T}{2} + \delta\right)$$

where δ is the phase difference between c_1 and c_2 $(B = 0)$.

In practice, the time spent in the B≠0 region, is fixed but the precession frequency to is varied by changing the strength of the magnetic field.

The intensity in the interference region as a function of B is predicted to have a sinosoidal variation.

If we call ΔB the difference in B needed to produce successive maxima, we can easily show that

$$\Delta B = \frac{4\pi\hbar c}{e g_n \lambda l}$$

where l is the path length.

$$T = \frac{4\pi}{\omega}$$

$$B = \frac{4\pi m_p c}{g_n e T}$$

$$\omega = \frac{g_n e B}{m_p c}$$

$$B = \frac{4\pi \hbar c}{g_n e \bar{\lambda} v T}$$

$$p = \frac{h}{\lambda} = \frac{\hbar}{\bar{\lambda}}$$

$$m_p = \frac{\hbar}{\bar{\lambda} v}$$

$$B = \frac{4\pi \hbar c}{g_n e \bar{\lambda} l}$$

In deriving this formula we used the fact that a 4π rotation is needed for the state ket to return to the original ket with the same sign, as required by our formalism. If, on the other hand, our description of spin $\frac{1}{2}$ systems were incorrect and the ket were to return to its original ket with the same sign under a 2π rotation, the predicted value for ΔB would be just one-half of (3.2.25).

Two different groups have conclusively demonstrated experimentally that prediction (3.2.25) is correct to an accuracy of a fraction of a percent.* This is another triumph of quantum mechanics. The nontrivial prediction (3.2.15) has been experimentally established in a direct manner.

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*H. Rauch et al., *Phys. Lett.* **54A**, 425 (1975); S. A. Werner et al., *Phys. Rev. Lett.* **35** (1975), 1053.

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Observation of the Phase Shift of a Neutron Due to Precession in a Magnetic Field*

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We have directly observed the sign reversal of the wave function of a fermion produced by its precession of 2π radians in a magnetic field using a neutron interferometer.

It is well known that the operator for rotation through 2π radians for a fermion causes a reversal of the sign of the wave function. We have directly observed this effect for neutrons precessing in a magnetic field using an interferometer of the type first developed for x rays by Bonse and Hart.[1] This experiment was first suggested by Bernstein[2] in 1967. At nearly the same time the possibility of observing this effect was noted by Aharonov and Susskind[3] and a tunneling experiment using electrons was proposed.

The interferometer, He^3 detectors, and peripheral apparatus employed in this experiment are the same as those used in the recent observation of gravitationally induced quantum interference.[4] A monoenergetic, unpolarized neutron beam ($\lambda = 1.445$ Å) is split at point A of the interferometer by Bragg reflection (Fig. 1). The one beam passes through a transverse dc magnetic field on the path AC. The relative phase of the two beams where they recombine and interfere at point D is varied by adjusting the magnetic field $\vec{B}$.

If we take the Hamiltonian for the neutron of momentum $\vec{p}$ and magnetic moment $\vec{\mu}$ to be

$$H \doteq p^2/2M - \vec{\mu}\cdot\vec{B}, \tag{1}$$

it is easy to show that the phase shift to first order in $\vec{B}$ is

$$\beta = \pm 2\pi g_n \mu_N M \lambda B l / h^2 . \tag{2}$$

Here the $\pm$ signs are for spin-up and spin-down neutrons; g_n is the neutron magnetic moment in nuclear magnetons ($= -1.91$), μ_N is the nuclear magneton, h is Planck's constant, M is the neu-

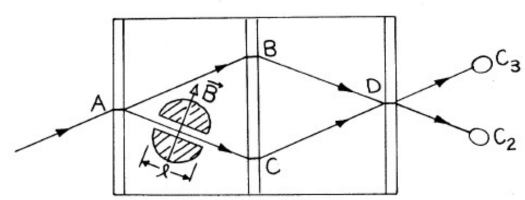

FIG. 1. A schematic diagram of the neutron interferometer. On the path AC the neutrons are in a magnetic field B (0 to 500 G) for a distance l (2 cm).

tron mass, and l is the distance over which the neutron is in the magnetic field.

We must add the contributions of spin-up and spin-down neutrons together since the experiment was done with unpolarized neutrons. There is always a residual phase shift δ in the interferometer due to various causes, including gravity. The counting rates at detectors C_2 and C_3 are expected to be

$$\begin{aligned} I_2 &= I_2(\uparrow) + I_2(\downarrow) \\ &= [\tfrac{1}{2}\gamma - \tfrac{1}{2}\alpha\cos(\delta+\beta)] + [\tfrac{1}{2}\gamma - \tfrac{1}{2}\alpha\cos(\delta-\beta)] \\ &= \gamma - \alpha\cos\delta\cos\beta, \end{aligned} \tag{3}$$

and

$$\begin{aligned} I_3 &= I_3(\uparrow) + I_3(\downarrow) \\ &= \tfrac{1}{2}\alpha[1+\cos(\delta+\beta)] + \tfrac{1}{2}\alpha[1+\cos(\delta-\beta)] \\ &= \alpha[1+\cos\delta\cos\beta]. \end{aligned} \tag{4}$$

In these expressions we have taken β to be defined by Eq. (2) with the plus sign. The constants α and γ are the same instrumental parameters of Ref. 4. Thus, if the residual phase δ is fortuitously $\pi/2$, $3\pi/2$, ..., there will be no observable effect of the magnetic field on the intensities.[5] We circumvent this problem by first rotating the interferometer about the incident beam AB, thus using the effect of gravity to set the phase at a minimum of $I_2 - I_3$ which insures that $\delta = 0$, 2π, ... The major problem in this experiment was finding a method for producing a variable magnetic field (~0 to 500 G) of uniform intensity over the beam dimensions (2 mm×10 mm) in a limited space, which does not disturb the interferometer by heating, or in any other way. Our solution to this problem was to construct a small magnet using two cobalt-samarium permanent magnets, one of which has a variable position as shown in Fig. 2.

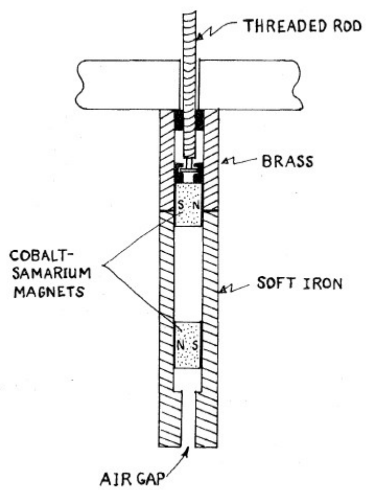

FIG. 2. Diagram of the magnet used in this experiment.

Equation (2) predicts that the field required for a precession of 4π (complete period) is

$$Bl = 272/\lambda, \tag{5}$$

where B is in gauss, l in centimeters, λ in angstroms. It is clear that the leakage field from the magnet must be included in a comparison of experiment with theory. We have experimentally determined $\vec{B}$ with a small magnetic field probe along the two beam paths ABD and ACD. The effective Bl for our magnet and interferometer is

$$\langle Bl\rangle = 2.7B_{\text{gap}}(\text{G cm}), \tag{6}$$

where B_{gap} is the magnetic field in the magnet air gap.

Our first results are shown in Fig. 3. The oscillation period is 62 ± 2 G. Thus,

$$\langle Bl\rangle = 242/\lambda. \tag{7}$$

The agreement of this result is within the experimental errors which we are willing to assign to the measurement of the effective Bl.

It is clear that we have observed the complete rotation symmetry demanded by the spinor character of the neutron wave function. In classical physics $2n\pi$ rotations are unobservable. It should also be noted that in a superconducting quantum interference experiment this effect is not observ-

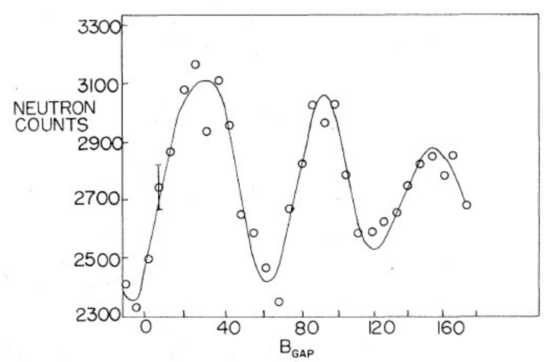

FIG. 3. The difference count, $I_2 - I_3$, as a function of the magnetic field in the magnet air gap in gauss. Approximate counting time was 40 min per point.

able because the charge carriers are bosons.

This experiment was carried out at the Ford Nuclear Reactor, University of Michigan. The technical assistance of B. Poindexter is gratefully acknowledged.

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[5]The importance of the residual phase δ was pointed out recently by M. A. Horne and A. Shimony, to be published.