خاصیت بنیادی عملگر وارونی زمان چیست؟

آیا عملگر وارونی زمان می تواند روی «براها» اثر کند یا فقط بایدروی «کت ها» اثر کند؟

آیا عملگر وارونی زمان با عملگرهای مکان و اندازه حرکت خط و اندازه حرکت زاویه ای جا به جا می شود؟

آیا عملگر وارونی زمان جابه جاگر اندازه حرکت و مکان را تغییر می دهد؟

Time-Reversal Operator

Consider

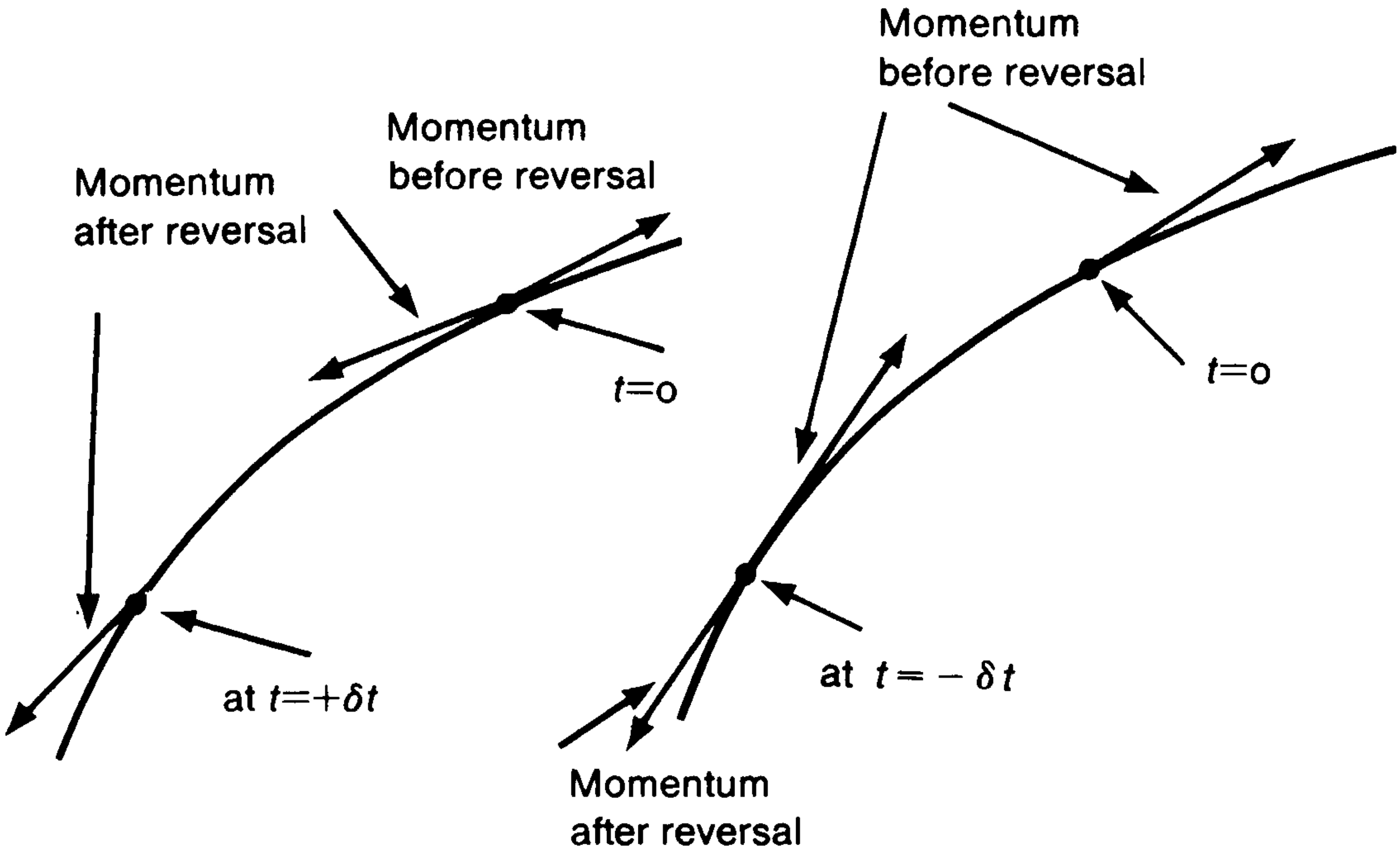
$$|\alpha\rangle \rightarrow \Theta|\alpha\rangle$$

where $\Theta|\alpha\rangle$ is the time-reversed state.

More appropriately, $\Theta|\alpha\rangle$ should be called the motion-reversed state.

If $|\alpha\rangle$ is a momentum eigenstate $|\mathbf{p}'\rangle$, we expect $\Theta|\alpha\rangle$ to be $|-\mathbf{p}'\rangle$ up to a possible phase.

Likewise, $\mathbf{J}$ is to be reversed under time reversal.



$$\left(1-\frac{iH}{\hbar}\delta t\right)\Theta|\alpha\rangle=\Theta\left(1-\frac{iH}{\hbar}(-\delta t)\right)|\alpha\rangle$$

$$-iH\Theta|\rangle=\Theta iH|\rangle$$

where the blank ket $|\rangle$ represents any ket.

We now argue that Θ cannot be unitary.

Suppose Θ were unitary.

It would then be legitimate to cancel the $i's$ in $-iH\Theta| >= \Theta iH| >$, and we would have the operator equation $-H\Theta = \Theta H$.

$$H\Theta|n\rangle = -\Theta H|n\rangle = (-E_n)\Theta|n\rangle$$

This equation says that $\Theta|n\rangle$ is an eigenket of the Hamiltonian with energy eigenvalues $-E_n$.

But this is nonsensical even in the very elementary case of a free particle.

We know that the energy spectrum of the free particle is positive semidefinite—from 0 to $+\infty$.

We expect p to change sign but not $\mathbf{p}^2$; yet $-H\Theta = \Theta H$ would imply that

$$\Theta^{-1}\frac{\mathbf{p}^2}{2m}\Theta = \frac{-\mathbf{p}^2}{2m}$$

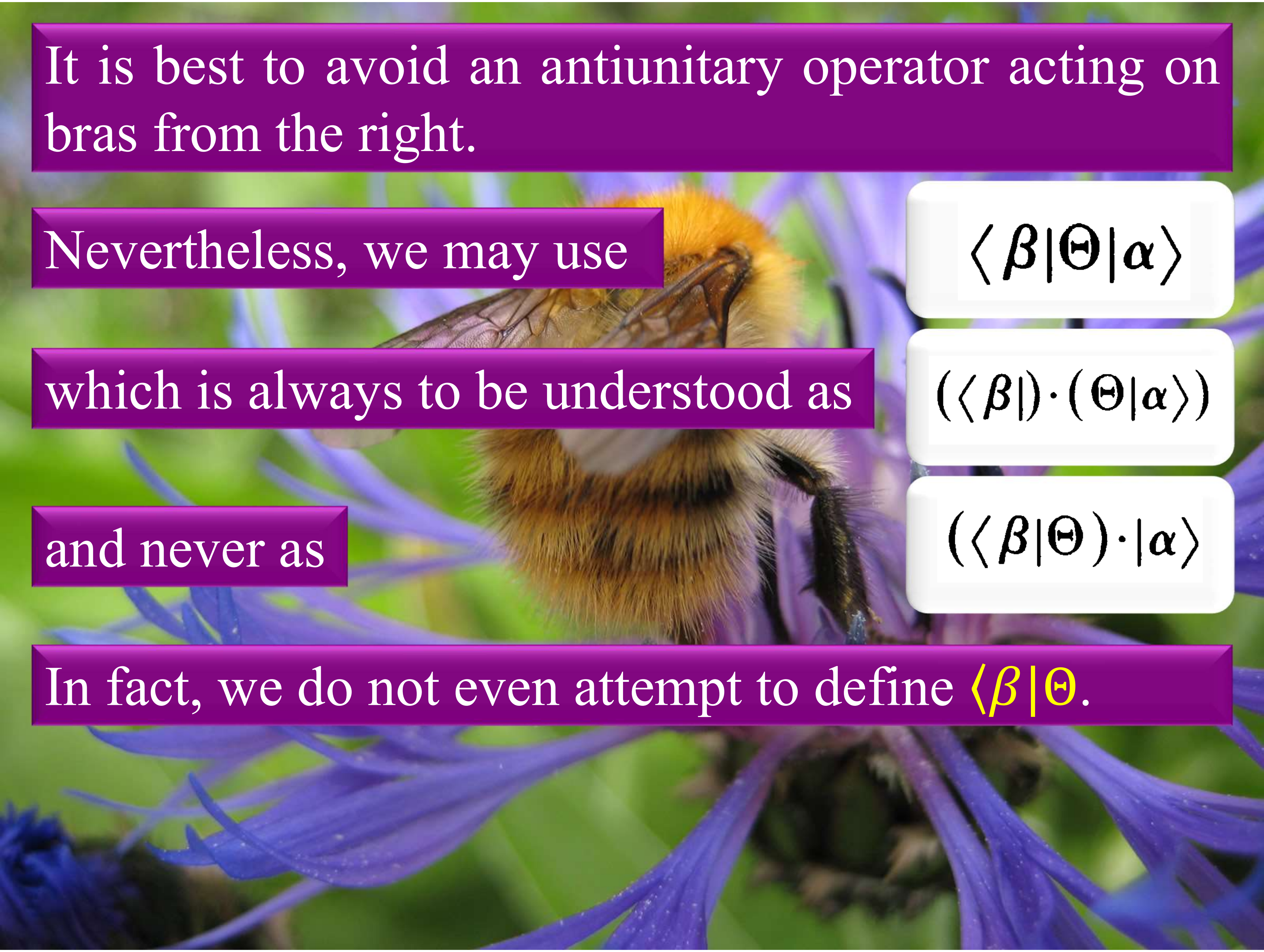
All these arguments strongly suggest that we are not allowed to cancel the $i's$ in $-iH\Theta| >= \Theta iH| >$; hence, had Θ better be antiunitary.

In this case: $\Theta iH| > = -i\Theta H| >$.

Now at last we can cancel the $i's$ in $-iH\Theta| > = \Theta iH| > = -i\Theta H| >$. Thus $\boldsymbol{\Theta H = H\Theta}$.

Equation $\boldsymbol{\Theta H = H\Theta}$ expresses the fundamental property of the Hamiltonian under time reversal.

With $\boldsymbol{\Theta H = H\Theta}$ the difficulties mentioned earlier [see $-H\Theta = \Theta H$, $H\Theta|n\rangle = -\Theta H|n\rangle = -(E_n)\Theta|n\rangle$ and $\Theta^{-1}\frac{\mathbf{p}^2}{2m}\Theta = -\frac{\mathbf{p}^2}{2m}$] are absent, and we obtain physically sensible results.



It is best to avoid an antiunitary operator acting on bras from the right.

Nevertheless, we may use

$$\langle \beta | \Theta | \alpha \rangle$$

which is always to be understood as

$$\left(\langle \beta | \right) \cdot \left(\Theta | \alpha \rangle \right)$$

and never as

$$\left(\langle \beta | \Theta \right) \cdot | \alpha \rangle$$

In fact, we do not even attempt to define $\langle \beta | \Theta$.

We start with an important identity, namely,

$$\langle \beta | \otimes | \alpha \rangle = \langle \tilde{\alpha} | \Theta \otimes^{\dagger} \Theta^{-1} | \tilde{\beta} \rangle$$

where $\otimes$ is a linear operator.

This identity follows solely from the antiunitary nature of Θ.

To prove this let us define

$$|\gamma\rangle \equiv \otimes^{\dagger} | \beta \rangle$$

By dual correspondence we have

$$|\gamma\rangle \overset{\mathrm{DC}}{\leftrightarrow} \langle \beta | \otimes = \langle \gamma |$$

Hence,

$$\begin{aligned}\langle\beta|\otimes|\alpha\rangle &= \langle\gamma|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\gamma}\rangle \\ &= \langle\tilde{\alpha}|\Theta\otimes^{\dagger}|\beta\rangle = \langle\tilde{\alpha}|\Theta\otimes^{\dagger}\Theta^{-1}\Theta|\beta\rangle \\ &= \langle\tilde{\alpha}|\Theta\otimes^{\dagger}\Theta^{-1}|\tilde{\beta}\rangle,\end{aligned}$$

which proves the identity.

In particular, for Hermitian observables A we get

$$\langle\beta|A|\alpha\rangle = \langle\tilde{\alpha}|\Theta A\Theta^{-1}|\tilde{\beta}\rangle$$

We say that observables are even or odd under time reversal according to whether we have the upper or lower sign in $\Theta A\Theta^{-1} = \pm A$.

Note that $\Theta A \Theta^{-1} = \pm A$ equation, together with $\langle \beta | A | \alpha \rangle = \langle \tilde{\alpha} | \Theta A \Theta^{-1} | \tilde{\beta} \rangle$, gives a phase restriction on the matrix elements of A taken with respect to time reversed states as follows:

$$\langle \beta | A | \alpha \rangle = \pm \langle \tilde{\beta} | A | \tilde{\alpha} \rangle^*$$

If $|\beta\rangle$ is identical to $|\alpha\rangle$, so that we are talking about expectation values, we have

$$\langle \alpha | A | \alpha \rangle = \pm \langle \tilde{\alpha} | A | \tilde{\alpha} \rangle$$

where $\langle \tilde{\alpha} | A | \tilde{\alpha} \rangle$ is the expectation value taken with respect to the time-reversed state.

As an example, let us look at the expectation value of **p**.

It is reasonable to expect that the expectation value of **p** taken with respect to the time-reversed state be of opposite sign.

Thus

$$\langle \alpha|\mathbf{p}|\alpha\rangle = -\langle \tilde{\alpha}|\mathbf{p}|\tilde{\alpha}\rangle$$

so we take p to be an odd operator, namely,

$$\Theta \mathbf{p} \Theta^{-1} = -\mathbf{p}$$

This implies that

$$\begin{aligned}\mathbf{p}\Theta|\mathbf{p}'\rangle &= -\Theta \mathbf{p} \Theta^{-1}\Theta|\mathbf{p}'\rangle \\ &= (-\mathbf{p}')\Theta|\mathbf{p}'\rangle.\end{aligned}$$

Equation $\mathbf{p}\Theta|\mathbf{p}'\rangle = -(\mathbf{p}')\Theta|\mathbf{p}'\rangle$ agrees with our earlier assertion that $\Theta|\mathbf{p}'\rangle$ is a momentum eigenket with eigenvalue $-\mathbf{p}'$.

It can be identified with $|-\mathbf{p}'\rangle$ itself with a suitable choice of phase.

Likewise, we obtain

$$\Theta \mathbf{x} \Theta^{-1} = \mathbf{x}$$

$$\Theta|\mathbf{x}'\rangle = |\mathbf{x}'\rangle \qquad \text{(up to a phase)}$$

from the (eminently reasonable) requirement

$$\langle \alpha|\mathbf{x}|\alpha\rangle = \langle \tilde{\alpha}|\mathbf{x}|\tilde{\alpha}\rangle$$

We can now check the invariance of the fundamental commutation relation

$$[x_i, p_j]|\,\rangle = i\hbar\delta_{ij}|\,\rangle$$

where the blank ket |> stands for any ket.

Applying Θ to both sides of the above equation, we have

$$\Theta[x_i, p_j]\Theta^{-1}\Theta|\,\rangle = \Theta i\hbar\delta_{ij}|\,\rangle$$

which leads to, after passing Θ through $i\hbar$.

$$[x_i,(-p_j)]\Theta|\,\rangle = -i\hbar\delta_{ij}\Theta|\,\rangle$$

Note that the fundamental commutation relation $[x_i, p_j] = i\hbar\delta_{ij}$ is preserved by virtue of the fact that Θ is antiunitary.

This can be given as yet another reason for taking Θ to be antiunitary; otherwise, we will be forced to abandon either $\Theta\mathbf{p}\Theta^{-1} = -\mathbf{p}$ or $\Theta\mathbf{x}\Theta^{-1} = \mathbf{x}$!

Similarly, to preserve $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$ the angular-momentum operator must be odd under time reversal, that is,

$$\Theta\mathbf{J}\Theta^{-1} = -\mathbf{J}$$

This is consistent for spinless system where **J** is just **x** × **p**.

Alternatively, we could have deduced this relation by noting that the rotational operator and the time-reversal operator commute (note the extra *i*!).