TIME-DEPENDENT POTENTIALS: THE INTERACTION PICTURE

So far in this book we have been concerned with Hamiltonians that do not contain time explicitly.

In nature, however, there are many quantum-mechanical systems of importance with time dependence.

In the remaining part of this chapter we show how to deal with situations with time-dependent potentials.

$$H = H_0 + V(t)$$

$H_0 \neq f(t)$

The problem V(t) = 0 is assumed to be solved exactly.

$$H_0|n\rangle = E_n|n\rangle$$

$$|\alpha\rangle = \sum_n c_n(0)|n\rangle$$

$c_n(t) = ?$

$$|\alpha, t_0 = 0; t\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar}|n\rangle$$

The Interaction Picture

$$|\alpha, t_0; t\rangle_I = e^{\imath H_0 t/\hbar}|\alpha, t_0; t\rangle_S$$

$$A_I \equiv e^{\imath H_0 t/\hbar} A_S e^{-\imath H_0 t/\hbar}$$

$$V_I = e^{\imath H_0 t/\hbar} V e^{-\imath H_0 t/\hbar}$$

$$|\alpha\rangle_H = e^{+\imath H t/\hbar}|\alpha, t_0 = 0; t\rangle_S$$

$$A_H = e^{\imath H t/\hbar} A_S e^{-\imath H t/\hbar}$$

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = i\hbar\frac{\partial}{\partial t}\left(e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S\right)$$
$$= -H_0e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S + e^{iH_0t/\hbar}(H_0+V)|\alpha, t_0; t\rangle_S$$
$$= e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S$$

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I$$

which is a Schrodinger-like equation with the total H replaced by V_I.

Similarly we can also derive the a Hisenberg-like equation with H replaced by H_0

$$\frac{dA_I}{dt} = \frac{1}{i\hbar}[A_I, H_0]$$

In many respects, the interaction picture, or *Dirac picture*, is intermediate between the Schrödinger picture and the Heisenberg picture; This should be evident from Table 5.2.

TABLE 5.2

	Heisenberg picture	Interaction picture	Schrödinger picture
State ket	No change	Evolution determined by V_I	Evolution determined by H
Observable	Evolution determined by H	Evolution determined by H_0	No change

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I$$

$$i\hbar\frac{\partial}{\partial t}\langle n|\alpha, t_0; t\rangle_I = \sum_m \langle n|V_I|m\rangle\langle m|\alpha, t_0; t\rangle_I$$

$$\langle n|e^{iH_0 t/\hbar}V(t)e^{-iH_0 t/\hbar}|m\rangle = V_{nm}(t)e^{i(E_n - E_m)t/\hbar}$$

$$c_n(t) = \langle n|\alpha, t_0; t\rangle_I$$

$$i\hbar\frac{d}{dt}c_n(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t)$$

$$\omega_{nm} \equiv \frac{(E_n - E_m)}{\hbar} = -\omega_{mn}$$

$$i\hbar \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12}e^{i\omega_{12}t} & & \cdots \\ V_{21}e^{i\omega_{21}t} & V_{22} & & \cdots \\ & & V_{33} & \cdots \\ \vdots & \vdots & & \ddots \\ \vdots & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \end{pmatrix}$$

$$i\hbar \frac{d}{dt} c_n(t) = \sum_m V_{nm} e^{i\omega_{nm}t} c_m(t)$$

$$\omega_{nm} \equiv \frac{(E_n - E_m)}{\hbar} = -\omega_{mn}$$

Time-Dependent Two-State Problems: Nuclear Magnetic Resonance, Masers, and So Forth

Exact soluble problems with time-dependent potentials are rather rare. In most cases we have to resort to perturbation expansion to solve the coupled differential equations (5.5.17), as we will discuss in the next section. There is, however, a problem of enormous practical importance, which can be solved exactly—a two-state problem with a sinusoidal oscillating potential.

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| \quad (E_2 > E_1)$$

$$V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|$$

$$V_{12} = V_{21}^* = \gamma e^{i\omega t} \qquad V_{11} = V_{22} = 0$$

$$c_1(0) = 1, \qquad c_2(0) = 0$$

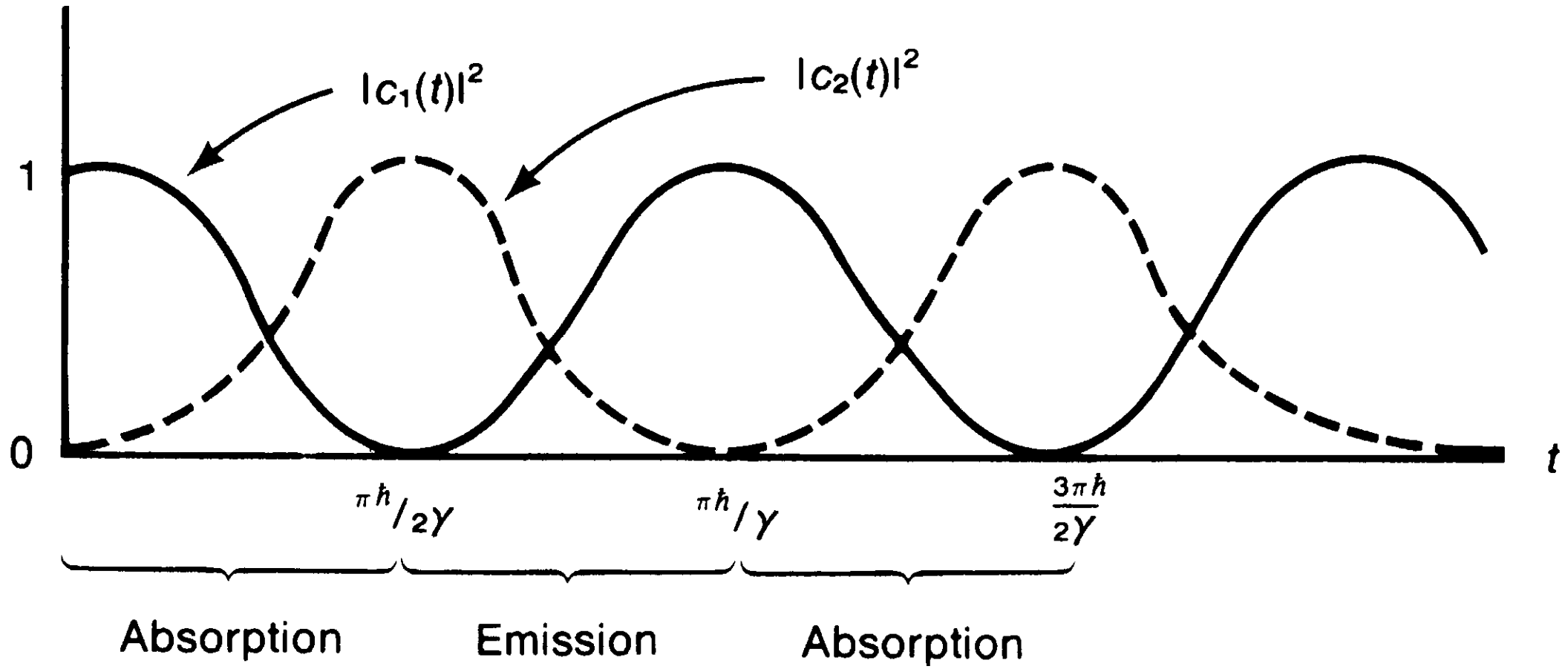
$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4}\right]^{1/2} t\right\}$$

$$\omega_{21} \equiv \frac{(E_2 - E_1)}{\hbar} \qquad \Omega = \sqrt{\left(\frac{\gamma^2}{\hbar^2}\right) + \frac{(\omega - \omega_{21})^2}{4}}$$

$$|c_1(t)|^2 = 1 - |c_2(t)|^2$$

Resonance

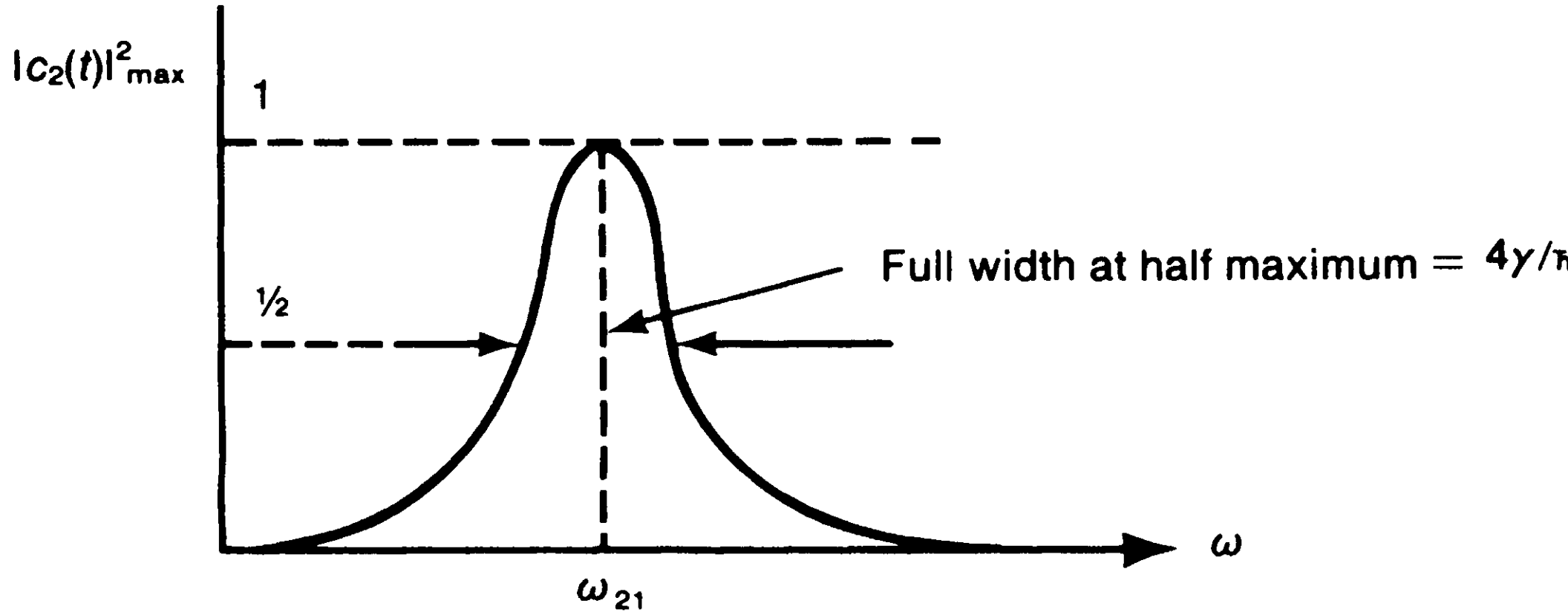
$$\omega \simeq \omega_{21} = \frac{(E_2 - E_1)}{\hbar} \qquad \omega = \omega_{21}, \qquad \Omega = \frac{\gamma}{\hbar}$$



The absorption-emission cycle takes place even away from resonance. However, the amplitude of oscillation for $|2\rangle$ is now reduced; $|c_2(t)|^2_{\max}$ is no longer 1 and $|c_1(t)|^2$ does not go down all the way to 0. In

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4}\right]^{1/2} t\right\}$$

$$|c_1(t)|^2 = 1 - |c_2(t)|^2$$



Spin Magnetic Resonance

$$\mathbf{B} = B_0\hat{\mathbf{z}} + B_1(\hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t)$$

$$\mu = \frac{e}{m_e c}\mathbf{S}$$

$$H_0 = -\left(\frac{e\hbar B_0}{2m_e c}\right)(|+\rangle\langle +| - |-\rangle\langle -|)$$

$$V(t) = -\left(\frac{e\hbar B_1}{2m_e c}\right)[\cos\omega t(|+\rangle\langle -| + |-\rangle\langle +|) + \sin\omega t(-i|+\rangle\langle -| + i|-\rangle\langle +|)],$$

$$S_z = (\hbar/2)[(|+\rangle\langle +|) - (|-\rangle\langle -|)]$$

$$S_x = \frac{\hbar}{2}[(|+\rangle\langle -|) + (|-\rangle\langle +|)],$$

$$S_y = \frac{\hbar}{2}[-i(|+\rangle\langle -|) + i(|-\rangle\langle +|)].$$

With e < 0, we have $E_+ > E_-$

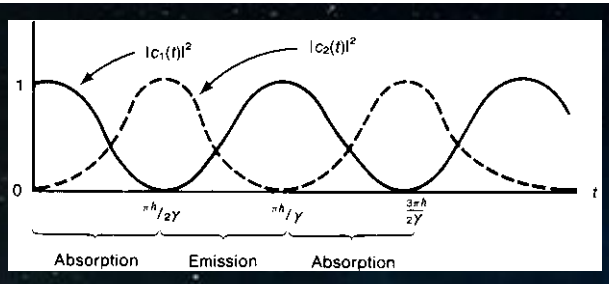
$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| \quad (E_2 > E_1)$$
$$V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|,$$

$|+\rangle \rightarrow |2\rangle$ (upper level)

$|-\rangle \rightarrow |1\rangle$ (lower level)

and so we can identify

$$\omega_{21} = \frac{|e|B_0}{m_e c}$$



$|+\rangle \rightarrow |2\rangle$ (upper level)

$|-\rangle \rightarrow |1\rangle$ (lower level)

which is just the spin-precession frequency for the $B_0 \neq 0$, $B_1 = 0$ problem already treated in Section 2.1. Even though the expectation values of $\langle S_{x,y} \rangle$ change due to spin precession in the counterclockwise direction (seen from the positive z-side), $|c_+|^2$ and $|c_-|^2$ remain unchanged in the absence of the rotating field. We now add a new feature as a result of the rotating field: $|c_+|^2$ and $|c_-|^2$ do change as a function of time. This can be seen by identifying

$$\frac{-e\hbar B_1}{2m_e c} \rightarrow \gamma, \qquad \omega \rightarrow \omega$$

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4}\right]^{1/2} t\right\}$$

$$V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1| \equiv$$

$$V(t) = -\left(\frac{e\hbar B_1}{2m_e c}\right)[\cos\omega t(|+\rangle\langle -| + |-\rangle\langle +|) + \sin\omega t(-i|+\rangle\langle -| + i|-\rangle\langle +|)],$$

to make correspondence to the notation of (5.5.18); our time-dependent interaction (5.5.28) is precisely of form (5.5.18). The fact that $|c_+(t)|^2$ and $|c_-(t)|^2$ vary in the manner indicated by Figure 5.4 for $\omega = \omega_{21}$ and the correspondence (5.5.29), for example, implies that the spin $\frac{1}{2}$ system undergoes a succession of spin-flops, $|+\rangle \rightleftarrows |-\rangle$, in addition to spin precession. Semiclassically, spin-flops of this kind can be interpreted as being due to the driving torque exerted by rotating **B**.

The resonance condition is satisfied whenever the frequency of the rotating magnetic field coincides with the frequency of spin precession determined by the strength of the uniform magnetic field. We see that the probability of spin-flops is particularly large.

In practice, a rotating magnetic field may be difficult to produce experimentally. Fortunately, a horizontally oscillating magnetic field—for instance, in the x-direction— is just as good. To see this, we first note that such an oscillating field can be decomposed into a counterclockwise component and a clockwise component as follows:

$$2B_1\hat{\mathbf{x}}\cos\omega t = B_1(\hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t) + B_1(\hat{\mathbf{x}}\cos\omega t - \hat{\mathbf{y}}\sin\omega t)$$

We can obtain the effect of the counterclockwise component simply by reversing the sign of ω. Suppose the resonance condition is met for the counterclockwise component

$$\omega \simeq \omega_{21}$$

Under a typical experimental condition,

$$\frac{B_1}{B_0} \ll 1, \qquad \frac{-e\hbar B_1}{2m_e c} \to \gamma, \qquad \omega \to \omega$$

which implies from (5.5.30) and (5.5.31) that

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega-\omega_{21})^2/4} \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_{21})^2}{4}\right]^{1/2} t\right\} \qquad \frac{\gamma}{\hbar} \ll \omega_{21}; \qquad \omega_{21} = \frac{|e|B_0}{m_e c}$$

as a result, whenever the resonance condition is met for the counterclockwise component, the effect of the clockwise component becomes completely negligible, since it amounts to $\omega \to -\omega$, and the amplitude becomes small in magnitude as well as very rapidly oscillating.

The resonance problem we have solved is of fundamental importance in interpreting atomic molecular beam and nuclear magnetic resonance experiments. By varying the frequency of oscillating field, it is possible to make a very precise measurement of magnetic moment. We have based our discussion on the solution to differential equations (5.5.17); this problem can also be solved, perhaps more elegantly, by introducing the *rotating axis representation* of Rabi, Schwinger, and Van Vleck.

$$i\hbar \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12}e^{i\omega_{12}t} & & \cdots \\ V_{21}e^{i\omega_{21}t} & V_{22} & & \cdots \\ & & V_{33} & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

Maser

As another application of the time-dependent two-state problem, let us consider a **maser**. Specifically, we consider an ammonia molecule NH_3, which—as we may recall from Section 4.2—has two parity eigenstates $|S\rangle$ and $|A\rangle$ lying close together such that $|A\rangle$ is slightly higher. Let $\boldsymbol{\mu}_{el}$ be the electric dipole operator of the molecule. From symmetry considerations we expect that $\boldsymbol{\mu}_{el}$ is proportional to $\mathbf{x}$, the position operator for the N atom. The basic interaction is like $-\boldsymbol{\mu}_{el}\cdot\mathbf{E}$, where for a maser $\mathbf{E}$ is a time-dependent electric field in a microwave cavity:

$$\mathbf{E} = |\mathbf{E}|_{\max}\hat{\mathbf{z}}\cos\omega t. \tag{5.5.36}$$

It is legitimate to ignore the spatial variation of $\mathbf{E}$ because the wavelength in the microwave region is far larger than molecular dimension. The frequency ω is tuned to the energy difference between $|A\rangle$ and $|S\rangle$:

$$\omega \simeq \frac{(E_A - E_S)}{\hbar}. \tag{5.5.37}$$

The diagonal matrix elements of the dipole operator vanish by parity:

$$\langle A|\boldsymbol{\mu}_{el}|A\rangle = \langle S|\boldsymbol{\mu}_{el}|S\rangle = 0, \tag{5.5.38}$$

but the off-diagonal elements are, in general, nonvanishing:

$$\langle S|\mathbf{x}|A\rangle = \langle A|\mathbf{x}|S\rangle \neq 0. \tag{5.5.39}$$

This means that there is a time-dependent potential that connects $|S\rangle$ and $|A\rangle$, and the general two-state problem we discussed earlier is now applicable.

We are now in a position to discuss how masers work. Given a molecular beam of NH_3 containing both $|S\rangle$ and $|A\rangle$, we first eliminate the $|S\rangle$ component by letting the beam go through a region of time-independent inhomogeneous electric field. Such an electric field separates $|S\rangle$ from $|A\rangle$ in much the same way as the inhomogeneous magnetic field in the Stern-Gerlach experiment separates $|+\rangle$ from $|-\rangle$. A pure beam of $|A\rangle$ then enters a microwave cavity tuned to the energy difference $E_A - E_S$. The dimension of the cavity is such that the time spent by the molecule is just $(\pi/2)\hbar/\gamma$. As a result we stay in the first emission phase of Figure 5.4; we have $|A\rangle$ in and $|S\rangle$ out. The excess energy of $|A\rangle$ is given up to the time-dependent potential as $|A\rangle$ turns into $|S\rangle$ and the radiation (microwave) field gains energy. In this way we obtain Microwave Amplification by Stimulated Emission of Radiation, or MASER.

There are many other applications of the general time-dependent two-state problem, such as the atomic clock and optical pumping. In fact, it is amusing to see that as many as four Nobel Prizes in physics have been awarded to those who exploited time-dependent two-state systems of some form.*