PERMUTATION SYMMETRY AND YOUNG TABLEAUX

In keeping track of the requirement imposed by permutation symmetry, there is an extremely convenient bookkeeping technique known as **Young tableaux**, after A. Young, an English clergyman who published a fundamental paper on this subject in 1901. This section is meant to be an introduction to Young tableaux for the practical-minded reader. We do not necessarily present all derivations; this is one of those cases where the rules are simpler than the derivations.

افراز یک عدد صحیح

نمودار یانگ

یک افراز $\lambda \equiv \{\lambda_1,\lambda_2, \dots ,\lambda_n\}$ از عدد صحیح n یک دنباله از اعداد صحیح و مثبت λ_i است، که

به ترتیب نزولی مرتب شده باشد و مجموع آنها n باشد، یعنی

$$\lambda_i \geq \lambda_{i+1}; i = 1, 2, \dots , r - 1$$

$$\sum_{i=1}^{r} \lambda_i = n$$

دو افراز λ و μ مساویند اگر

$$\forall i;\ \ \lambda_i = \mu_i$$

$\lambda > \mu$ است، اگر اولین عدد غیر صفر در دنباله $\lambda_i - \mu_i$ مثبت باشد

$\lambda < \mu$ است، اگر اولین عدد غیر صفر در دنباله $\lambda_i - \mu_i$ منفی باشد

یک افراز λ با نمودار یانگ به طور ترسیمی

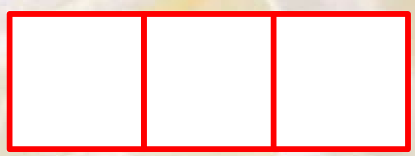
نمایش داده می شود. این نمودار شامل n مربع در r سطر است.

i–امین سطر شامل λ_i مربع است.

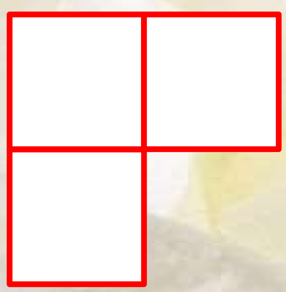
مثال

$n = 3$

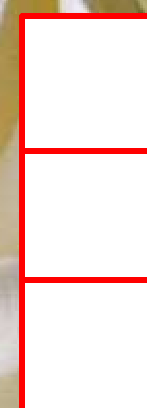
$\lambda = \{3, 0, 0\}$



$\mu = \{2, 1, 0\}$

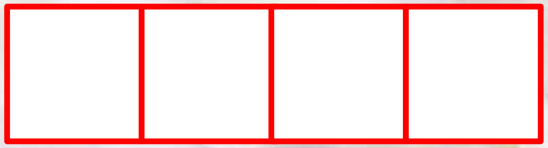


$\nu = \{1, 1, 1\}$

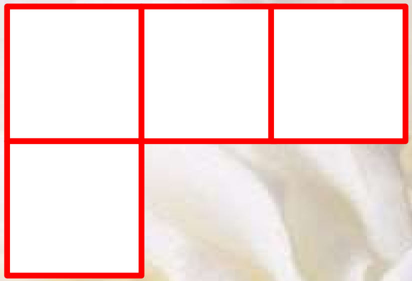


$n = 4$

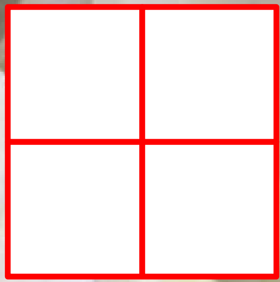
$\lambda = \{4, 0, 0, 0\}$



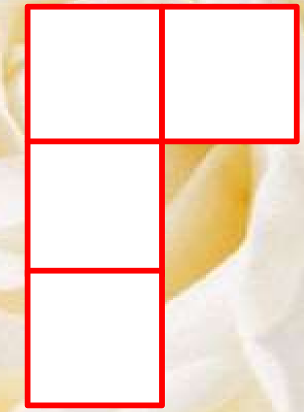
$\mu = \{3, 1, 0, 0\}$



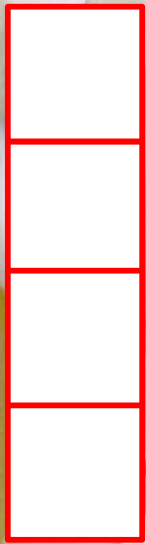
$\nu = \{2, 2, 0, 0\}$



$\xi = \{2, 1, 1, 0\}$

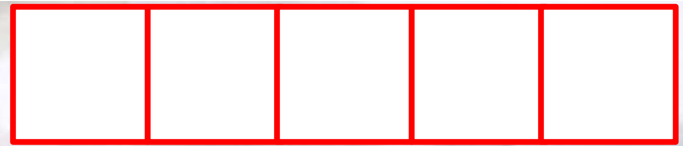


$\tau = \{1, 1, 1, 1\}$

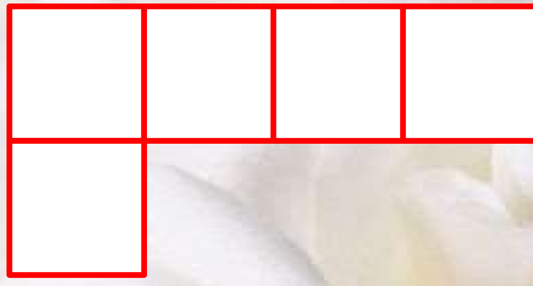


$n = 5$

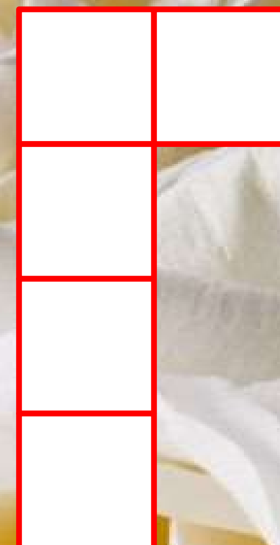
$\lambda \ = \{5, 0, 0,0, 0\}$



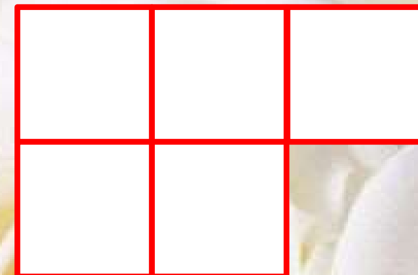
$\mu = \{4, 1, 0,0, 0\}$



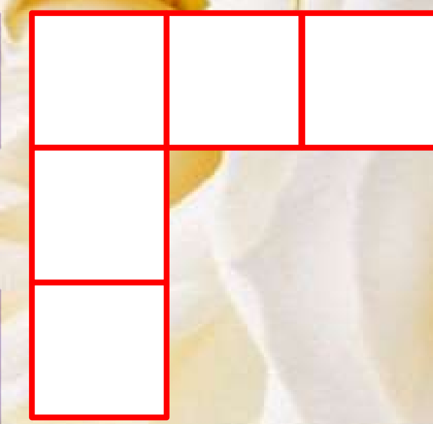
$\sigma = \{2, 1\ 1,1,0\}$



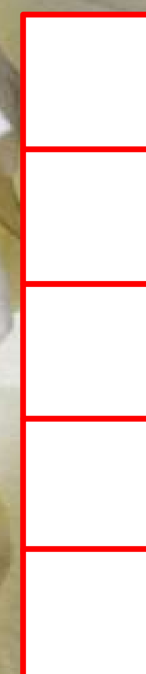
$\xi = \{3, 2, 0,0,0\}$



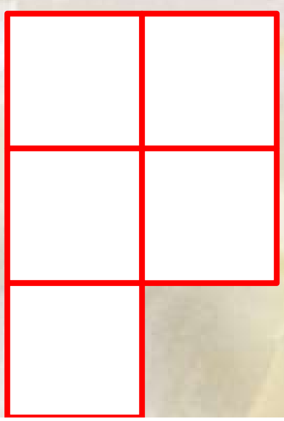
$\nu = \{3, 1, 1,0,0\}$



$\eta = \{1, 1\ 1,1,1\}$



$\tau = \{2, 2, 1,0,0\}$



تابلوی یانگ، تابلوی نرمال، تابلوی

تابلوی یانگ از پر کردن مربعهای یک نمودار یانگ با اعداد 1 تا n به هر ترتیب دلخواهی به دست می آید، مشروط بر آن که هر عدد فقط یک بار مورد استفاده قرار گیرد.

یک تابلوی نرمال یانگ تابلویی است که در آن اعداد 1 تا n به ترتیب از چپ به راست و از سطر بالا تا سطر پایین قرار گرفته باشد.

یک تابلوی استاندارد یانگ تابلویی است که در آن اعداد در هر سطر افزایشی (نه لزوما به ترتیب) از چپ به راست و در هر ستون آن نیز افزایشی از بالا به پایین باشد.

تابلوی نرمال

افزایشی

افزایشی

1	2
3	4

افزایشی

افزایشی

1	2	3
4		

افزایشی

پشت سرهم (به ترتیب)

تابلوی استاندارد

افزایشی

افزایشی

1	3
2	4

افزایشی

افزایشی

1	2	4
3		

افزایشی نیست

پشت سرهم (به ترتیب) نیستند

Θ_λ $\equiv$ یک تابلوی نرمال یانگ

Θ_λ^P $\equiv$ یک تابلوی دلخواه یانگ

$$\Theta_\lambda^P \equiv P\Theta_\lambda$$

یک تابلوی دلخواه یانگ از یک جایگشت مثلا P از یک تابلوی نرمال یانگ به دست می آید.

To illustrate the basic techniques involved, we consider once again the spin states of a two-electron system. We have three symmetric states corresponding to the three possible orientations of the spin-triplet state and an antisymmetric state corresponding to the spin singlet. The spin state of an *individual* electron is to be represented by a box. We use $\boxed{1}$ for spin up and $\boxed{2}$ for spin down. These boxes are the basic *primitive objects* of SU(2). A single box represents a doublet.

We define a symmetric tableau, $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$, and an antisymmetric tableau, $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$ When applied to the spin states of a two-electron system, $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$ is the Young tableau for a spin triplet, while $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$ is the Young tableau for a spin singlet. Returning now to $\boxed{1}$ and $\boxed{2}$, we can build up the three-triplet states as follows:

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 2 \\ \hline 2 & 2 \\ \hline\end{array}\right. . \tag{6.5.1}$$

We do not consider $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$ because when we put boxes horizontally, symmetry is understood. So we deduce an important rule: Double counting is avoided if we require that the number (label) *not decrease* going from the left to the right.

As for the antisymmetric spin-singlet state,

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \tag{6.5.2}$$

is the only possibility. Clearly $\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array}$ and $\begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline \end{array}$ are impossible because of the requirement of antisymmetry; for a vertical tableau we cannot have a symmetric state. Furthermore, $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$ is discarded to avoid double counting. To eliminate the unwanted symmetry states, we therefore require the number (label) to *increase* as we go down.

We take the following to be the general rule. In drawing Young tableau, going from left to right the number cannot decrease; going down the number must increase. We deduced this rule by considering the spin states of two electrons, but we can show this rule to be applicable to the construction of any tableau.

Consider now three electrons. We can construct totally symmetric spin states by the following rule:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array} \quad \left\{ \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 2 \\ \hline 1 & 2 & 2 \\ \hline 2 & 2 & 2 \\ \hline\end{array} \right. . \tag{6.5.3}$$

This method gives four states altogether. This is just the multiplicity of the $j = \frac{3}{2}$ state, which is obviously symmetric as seen from the $m = \frac{3}{2}$ case, where all three spins are aligned in the positive z-direction.

What about the totally antisymmetric states?

We may try vertical tableau like

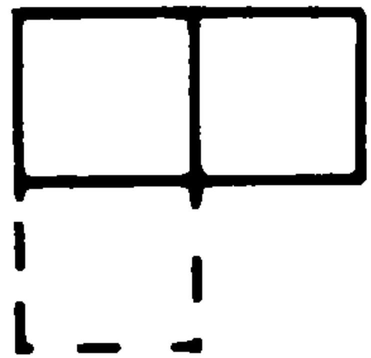
$$\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 2 \\ \hline \end{array}$$

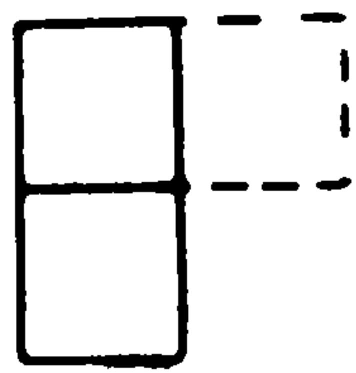
But these are illegal, because the numbers must increase as we go down. This is not surprising because total antisymmetry is impossible for spin states of three electrons; quite generally, a necessary (but, of course, not sufficient) condition for total antisymmetry is that every state must be different. In fact, in SU(2) we cannot have three boxes in a vertical column.

If $k'=k''$, then $|k',k'',k'''\rangle=0$ in

$$\begin{aligned}|k'k''k'''\rangle_{\pm} \equiv \frac{1}{\sqrt{6}}\{&|k'\rangle|k''\rangle|k'''\rangle \pm |k''\rangle|k'\rangle|k'''\rangle \\ &+ |k''\rangle|k'''\rangle|k'\rangle \pm |k'''\rangle|k''\rangle|k'\rangle \\ &+ |k'''\rangle|k'\rangle|k''\rangle \pm |k'\rangle|k'''\rangle|k''\rangle\}.\end{aligned}$$

We now define a mixed symmetry tableau that looks like this: . Such a tableau can be visualized as either a single box attached to a symmetric tableau [as in Equation (6.5.4a)] or a single box attached to an antisymmetric tableau [as in Equation (6.5.4b)].





If the spin function for three electrons is symmetric in two of the indices, neither of them can be antisymmetric with respect to a third index. For example,

$$(|+\rangle_1|-\rangle_2+|-\rangle_1|+\rangle_2)|-\rangle_3 \tag{6.5.5}$$

satisfies symmetry under $1 \leftrightarrow 2$, but it is neither symmetric nor antisymmetric with respect to $1 \leftrightarrow 3$ (or $2 \leftrightarrow 3$). We may try to enforce antisymmetry under $1 \leftrightarrow 3$ by subtracting the same thing from (6.5.5) with 1 and 3 interchanged:

$$\begin{aligned} &|+\rangle_1|-\rangle_2|-\rangle_3-|+\rangle_3|-\rangle_2|-\rangle_1+|-\rangle_1|+\rangle_2|-\rangle_3-|-\rangle_3|+\rangle_2|-\rangle_1 \\ &\quad = (|+\rangle_1|-\rangle_3-|-\rangle_1|+\rangle_3)|-\rangle_2. \end{aligned} \tag{6.5.6}$$

This satisfies antisymmetry under $1 \leftrightarrow 3$, but we no longer have the original symmetry under $1 \leftrightarrow 2$. In any case the dimensionality of $\yng(2,1)$ is 2; that is, it represents a doublet ($j = \frac{1}{2}$). We can see this by noting that such a tableau corresponds to the angular momentum addition of a spin doublet ($\yng(1)$) and a spin triplet ($\yng(2)$). We have used up the totally symmetric quartet $\yng(3)$, so the remainder must be a doublet. Alternatively, we may attach a doublet ($\yng(1)$) to a singlet ($\yng(1,1)$) as in (6.5.4b); the net product is obviously a doublet. So no matter how we consider it, $\yng(2,1)$ must represent a doublet. But this is precisely what the rule gives. If the number cannot decrease in the horizontal direction and must increase in the vertical direction, the only possibilities are

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}.$$

Because there are only two possibilities, $\yng(2,1)$ must correspond to a doublet. Notice that we do not consider $\ydiagram{2,1+1}$.

We can consider Clebsch-Gordan series or angular-momentum addition as follows:

$$\left\{\begin{array}{l} \mathscr{D}^{(1/2)}\otimes\mathscr{D}^{(1/2)} = \mathscr{D}^{(1)}\oplus\mathscr{D}^{(0)} \\ \ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \end{array}\right. \qquad (\mathbf{2}\times\mathbf{2}=\mathbf{3}+\mathbf{1})$$

$$\left\{\begin{array}{l} \mathscr{D}^{(1)}\otimes\mathscr{D}^{(1/2)} = \mathscr{D}^{(3/2)}\oplus\mathscr{D}^{(1/2)} \\ \ydiagram{2} \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1} \end{array}\right. \qquad (\mathbf{3}\times\mathbf{2}=\mathbf{4}+\mathbf{2}) \tag{6.5.8}$$

$$\left\{\begin{array}{l} \mathscr{D}^{(0)}\otimes\mathscr{D}^{(1/2)} = \mathscr{D}^{(1/2)} \\ \ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \end{array}\right. \qquad \left(\text{Note: } \ydiagram{1,1,1} \text{ is impossible; } \mathbf{1}\times\mathbf{2}=\mathbf{2}.\right)$$

We now extend our consideration to three primitive objects. A box can assume three possibilities:

$$\square : \boxed{1}, \boxed{2}, \boxed{3}. \tag{6.5.9}$$

The labels 1, 2, and 3 may stand for the magnetic quantum numbers of p-orbitals in atomic physics or charge states of the pion π^+, π^0, π^-, or the u, d, and s quarks in the SU(3) classification of elementary particles. We *assume* that the rule we inferred using two primitive objects can be generalized and work out such concepts as the dimensionality; we then check to see whether everything is reasonable.

$$\begin{array}{|c|}\hline \ \\\hline\end{array} : \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline\end{array} : \text{dimensionality } \mathbf{3}$$

Antisymmetry

$$\begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array} : \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \quad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} : \text{dimensionality } \mathbf{3}$$

(In SU(3) **3*** is used to distinguish $\begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array}$ from $\begin{array}{|c|}\hline \ \\\hline\end{array}$.)

$$\begin{array}{|c|}\hline \ \\\hline \ \\\hline \ \\\hline\end{array} : \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \quad \text{only one}$$

(Totally antisymmetrical.)

$$(6.5.10)$$

Symmetry

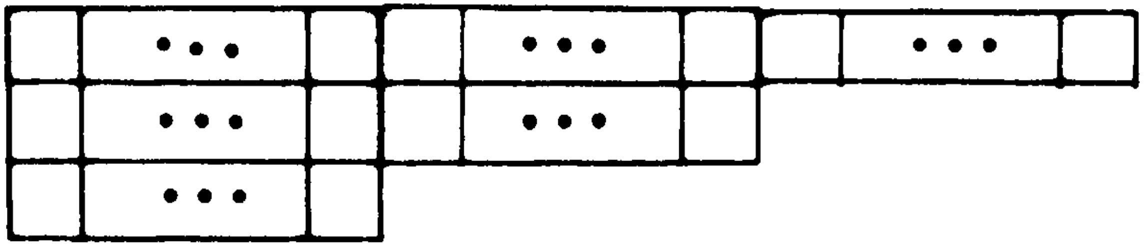
$$\begin{array}{|c|c|}\hline \ & \ \\\hline\end{array} : \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&3\\\hline\end{array} : \text{dimensionality } \mathbf{6}$$

$$\begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline\end{array} : \left.\begin{matrix} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \\ \begin{array}{|c|c|c|}\hline 1&3&3\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 2&2&3\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 2&3&3\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 3&3&3\\\hline\end{array} \end{matrix}\right\} : \text{dimensionality } \mathbf{10}$$

Mixed symmetry

$$\begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array} : \left.\begin{matrix} \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \\ \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \end{matrix}\right\} : \text{dimensionality } \mathbf{8}$$

These tableaux correspond to representations of SU(3).



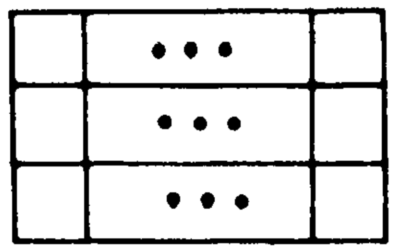
λ_1 boxes in first row

λ_2 boxes in second row

λ_3 boxes in third row

(6.5.11)

For the purpose of figuring the dimensionality, it is legitimate to strike out



(6.5.12)

a "singlet" at the left-hand side of (6.5.11). We can show the dimensionality to be

$$d(\lambda_1, \lambda_2, \lambda_3) = \frac{(p+1)(q+1)(p+q+2)}{2} \tag{6.5.13}$$

where $p = \lambda_1 - \lambda_2$ and $q = \lambda_2 - \lambda_3$.

In the SU(3) language, a tableau corresponds to a definite irreducible representation of SU(3). On the other hand, when each box is interpreted to mean a $j=1$ object, then a tableau does not correspond to an irreducible representation of the rotation group. Let us discuss this point more specifically. We start with putting together two boxes, where each box represents a $j=1$ state.

$$\square \otimes \square = \square\square \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}\,. \qquad (\mathbf{3} \times \mathbf{3} = \mathbf{6} + \mathbf{3}) \tag{6.5.14}$$

The horizontal tableau has six states; the tableau is to be broken down into $j=2$ (multiplicity 5) and $j=0$ (multiplicity 1), both of which are symmetric. In fact this is just how we construct a symmetric second-rank tensor out of two vectors. The vertical tableau $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$ corresponds to an antisymmetric $j=1$ state behaving like $\mathbf{a}\times\mathbf{b}$.

To work out the addition of three angular momenta with $j_1 = j_2 = j_3 = 1$, let us first figure out

$$
\begin{aligned}
&\ydiagram{2} \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1} \qquad (\mathbf{6} \times \mathbf{3} = \mathbf{10} + \mathbf{8}) \\
&\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1} \qquad (\mathbf{3} \times \mathbf{3} = \mathbf{8} + \mathbf{1})
\end{aligned}
\tag{6.5.15}
$$

To see how $\ydiagram{3}$ breaks down, we note that it must contain $j = 3$. However, this is not the only totally symmetric state; $\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) + \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ is also totally symmetric, and this has the transformation property of $j = 1$. So $\ydiagram{3}$ contains both $j = 3$ (seven states) and $j = 1$ (three states). As for $\ydiagram{2,1}$ with eight possibilities altogether, the argument is more involved, but we note that this $\mathbf{8}$ cannot be broken into $\mathbf{7} + \mathbf{1}$ because $\mathbf{7}$ is totally symmetric, while $\mathbf{1}$ is totally antisymmetric when we know that $\mathbf{8}$ is of *mixed* symmetry. So the only other possibility is $\mathbf{8} = \mathbf{5} + \mathbf{3}$—in other words $j = 2$ and $j = 1$.

Finally, therefore, we must have

$$\begin{array}{ccccccccc}
\yng(1)\otimes\yng(1)\otimes\yng(1) & = & \yng(3) & \oplus & \yng(2,1) & \oplus & \yng(2,1) & \oplus & \yng(1,1,1) \\
\mathbf{3}\times\mathbf{3}\quad\times\mathbf{3} & = & \underset{\substack{\uparrow \\ 7+3}}{\mathbf{10}} & + & \underset{\substack{\uparrow \\ 5+3}}{\mathbf{8}} & + & \underset{\substack{\uparrow \\ 5+3}}{\mathbf{8}} & + & \mathbf{1}
\end{array} \tag{6.5.16}$$

In terms of angular-momentum states, we have

$$\begin{array}{lll}
j=3 \text{ (dimension 7)} & \text{once} & \text{(totally symmetric)} \\
j=2 \text{ (dimension 5)} & \text{twice} & \text{(both mixed symmetry)} \\
j=1 \text{ (dimension 3)} & \text{three times} & \text{(one totally symmetric,} \\
 & & \quad \text{two mixed symmetry)} \\
j=0 \text{ (dimension 1)} & \text{once} & \text{(totally antisymmetric)}
\end{array} \tag{6.5.17}$$

Finally, we consider applications to elementary particle physics. Here we apply SU(3) considerations to the nonrelativistic quark model, where the primitive objects are u, d, s (up, down, and strange, where *up* and *down* refer to *isospin up* and *isospin down*). A box can now stand for u, d, or s. We look at the decuplet representation $\square\square\square$ (**10**):

$$
\begin{array}{lll}
\Delta^{++,+,0,-} & ddd\,udd\,uud\,uuu & I=\frac{3}{2}, \\
\Sigma^{+,0,-} & dds\,uds\,uus & I=1, \\
\Xi^{0,-} & dss\,uss & I=\frac{1}{2}, \\
\Omega^{-} & sss & I=0,
\end{array}
\tag{6.5.21}
$$

where I stands for isospin.

Now all ten states are known to be spin $\frac{3}{2}$ objects. It is safe to assume that the space part is in a relative S-state for low-lying states of three quarks. We expect total symmetry in the spin degree of freedom. For instance, the $j = \frac{3}{2}$, $m = \frac{3}{2}$ state of Δ can be visualized to have quark spins all aligned.

But the quarks are spin $\frac{1}{2}$ objects; we expect total antisymmetry because of Fermi-Dirac statistics. Yet

$$\begin{array}{ll} \text{Quark label (now called flavor):} & \text{symmetric} \\ \text{Spin:} & \text{symmetric} \\ \text{Space:} & \text{symmetric} \end{array} \qquad (6.5.22)$$

for the $j = \frac{3}{2}$ decuplet. But as is evident from (6.5.22), the total symmetry in this case is even! This led to the "statistics paradox," which was especially embarrassing because other aspects of the nonrelativistic quark model were so successful.

The way out of this dilemma is to postulate that there is actually an additional degree of freedom called *color* (red, blue, or yellow) and postulate that the observed hadrons (strongly interacting particles, including $J = \frac{3}{2}^+$ states considered here) are **color singlets**

$$\frac{1}{\sqrt{6}}(|\mathrm{RBY}\rangle - |\mathrm{BRY}\rangle + |\mathrm{BYR}\rangle - |\mathrm{YBR}\rangle + |\mathrm{YRB}\rangle - |\mathrm{RYB}\rangle). \qquad (6.5.23)$$

This is in complete analogy with the unique

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

of (6.5.10), a totally antisymmetric combination in color space. The statistics problem is now solved because

$$\underset{(-)}{P_{ij}} = \underset{(+)}{P_{ij}^{(\text{flavor})}} \; \underset{(+)}{P_{ij}^{(\text{spin})}} \; \underset{(+)}{P_{ij}^{(\text{space})}} \; \underset{(-)}{P_{ij}^{(\text{color})}} \qquad (6.5.24)$$

One might argue that this is a cheap way to get out of the dilemma. Fortunately, there were also other pieces of evidence in favor of color, such as the decay rate of π^0 and the cross section for electron-positron annihilation into hadrons. In fact, this is a very good example of how attempts to overcome difficulties lead to nontrivial prediction of color.

$$|2\ 0> \ = \ C_1|2\ \tfrac{3}{2}>|0\ -\tfrac{3}{2}> + C_2|2\ \tfrac{1}{2}>|0\ -\tfrac{1}{2}> + C_3|2\ -\tfrac{1}{2}>|0\ \tfrac{1}{2}> + C_4|2\ -\tfrac{3}{2}>|0\ \tfrac{3}{2}>$$

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

$$|2\ 0> \ = \ \frac{1}{\sqrt{4}}|2\ \tfrac{3}{2}>|0\ -\tfrac{3}{2}> + \frac{1}{\sqrt{4}}|2\ \tfrac{1}{2}>|0\ -\tfrac{1}{2}> - \frac{1}{\sqrt{4}}|2\ -\tfrac{1}{2}>|0\ \tfrac{1}{2}> - \frac{1}{\sqrt{4}}|2\ -\tfrac{3}{2}>|0\ \tfrac{3}{2}>$$

$$|j,m> = \sum_{m_1,m_2} C^{j,m}_{j_1,j_2;m_1,m_2}|j_1,m_1>|j_2,m_2>$$

$$C^{j,m}_{j_1,j_2;m_1,m_2} \ = \ \langle j_1,j_2;m_1,m_2|j_1,j_2;j,m\rangle$$

The Clebsch-Gordan coefficients are the solutions to

$$|(j_1 j_2) jm\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1 j_2 m_2\rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle$$

Explicitly:

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle =$$
$$\delta_{m,m_1+m_2} \sqrt{\frac{(2j+1)(j+j_1-j_2)!(j-j_1+j_2)!(j_1+j_2-j)!}{(j_1+j_2+j+1)!}} \times$$
$$\sqrt{(j+m)!(j-m)!(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!} \times$$
$$\sum_k \frac{(-1)^k}{k!(j_1+j_2-j-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j-j_2+m_1+k)!(j-j_1-m_2+k)!}.$$

The summation is extended over all integral *k* for which the argument of every factorial is nonnegative.[4]

For brevity, solutions with m < 0 and $j_1 < j_2$ are omitted. They may be calculated using the simple relations

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle = (-1)^{j-j_1-j_2} \langle j_1 j_2; -m_1, -m_2 | j_1 j_2; j, -m\rangle.$$

and

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle = (-1)^{j_1+j_2-j} \langle j_2 m_2 j_1 m_1 | j_1 j_2; jm\rangle.$$

j_2=0

When j_2 = 0, the Clebsch-Gordan coefficients are given by $\delta_{j,j_1}\delta_{m,m_1}$.

j_1=1/2, j_2=1/2

m=1 j=

m₁, m₂=

	1
1/2, 1/2	1

m=0 j=

m₁, m₂=

	1	**0**
1/2, -1/2	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
-1/2, 1/2	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

$j_1=1$, $j_2=1/2$

m=3/2 j=

m₁, m₂=

	3/2
1, 1/2	1

m=1/2 j=

m₁, m₂=

	3/2	1/2
1, -1/2	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
0, 1/2	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$

$j_1=1,\ j_2=1$

$m=2$

$m_1, m_2=$ \ $j=$	2
1, 1	1

$m=1$

$m_1, m_2=$ \ $j=$	2	1
1, 0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
0, 1	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

$m=0$

$m_1, m_2=$ \ $j=$	2	1	0
1, -1	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$
0, 0	$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$
-1, 1	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$

$j_1=3/2,\ j_2=1/2$

$m=2$ $j=$

$m_1, m_2=$

	2
3/2, 1/2	1

$m=1$ $j=$

$m_1, m_2=$

	2	1
3/2, -1/2	$\frac{1}{2}$	$\sqrt{\frac{3}{4}}$
1/2, 1/2	$\sqrt{\frac{3}{4}}$	$-\frac{1}{2}$

$m=0$ $j=$

$m_1, m_2=$

	2	1
1/2, -1/2	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
-1/2, 1/2	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

$j_1=3/2$, $j_2=1$

m=5/2 j=

m₁, m₂=

	5/2
3/2, 1	1

m=3/2 j=

m₁, m₂=

	5/2	3/2
3/2, 0	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
1/2, 1	$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$

m=1/2 j=

m₁, m₂=

	5/2	3/2	1/2
3/2, -1	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{1}{2}}$
1/2, 0	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{1}{3}}$
-1/2, 1	$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$

$j_1=3/2$, $j_2=3/2$

$m=3$ $j=$

$m_1, m_2=$

	3
3/2, 3/2	1

$m=2$ $j=$

$m_1, m_2=$

	3	2
3/2, 1/2	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
1/2, 3/2	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

$m=1$ $j=$

$m_1, m_2=$

	3	2	1
3/2, -1/2	$\sqrt{\frac{1}{5}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{3}{10}}$
1/2, 1/2	$\sqrt{\frac{3}{5}}$	0	$-\sqrt{\frac{2}{5}}$
-1/2, 3/2	$\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{3}{10}}$

$m=0$ $j=$

$m_1, m_2=$

	3	2	1	0
3/2, -3/2	$\sqrt{\frac{1}{20}}$	$\frac{1}{2}$	$\sqrt{\frac{9}{20}}$	$\frac{1}{2}$
1/2, -1/2	$\sqrt{\frac{9}{20}}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{20}}$	$-\frac{1}{2}$
-1/2, 1/2	$\sqrt{\frac{9}{20}}$	$-\frac{1}{2}$	$-\sqrt{\frac{1}{20}}$	$\frac{1}{2}$
-3/2, 3/2	$\sqrt{\frac{1}{20}}$	$-\frac{1}{2}$	$\sqrt{\frac{9}{20}}$	$-\frac{1}{2}$