7.3 The Born approximation

Equation (7.33) is not particularly useful, as it stands, because the quantity $f(\mathbf{k}', \mathbf{k})$ depends on the unknown ket $|\psi\rangle$. Recall that $\psi(\mathbf{r}) = \langle \mathbf{r}|\psi\rangle$ is the solution of the integral equation

$$\psi(\mathbf{r}) = \phi(\mathbf{r}) - \frac{m}{2\pi\,\hbar^2}\frac{\exp(\mathrm{i}\,k\,r)}{r}\int \exp(-\mathrm{i}\,\mathbf{k}'\cdot\mathbf{r}')\,V(\mathbf{r}')\,\psi(\mathbf{r}')\,d^3\mathbf{r}', \tag{7.34}$$

where $\phi(\mathbf{r})$ is the wave-function of the incident state. According to the above equation the total wave-function is a superposition of the incident wave-function and lots of spherical-waves emitted from the scattering region. The strength of the spherical-wave emitted at a given point is proportional to the local value of the scattering potential, V, as well as the local value of the wave-function, ψ.

$$\frac{d\sigma}{d\Omega} = |f(\mathbf{k}', \mathbf{k})|^2. \tag{7.33}$$

$$\begin{aligned} f(\mathbf{k}', \mathbf{k}) &= -\frac{(2\pi)^2\,m}{\hbar^2}\int \frac{\exp(-\mathrm{i}\,\mathbf{k}'\cdot\mathbf{r}')}{(2\pi)^{3/2}}\,V(\mathbf{r}')\,\psi(\mathbf{r}')\,d^3\mathbf{r}' \\ &= -\frac{(2\pi)^2\,m}{\hbar^2}\,\langle \mathbf{k}'|H_1|\psi\rangle. \end{aligned}$$

Suppose that the scattering is not particularly strong. In this case, it is reasonable to suppose that the total wave-function, $\psi(\mathbf{r})$, does not differ substantially from the incident wave-function, $\phi(\mathbf{r})$. Thus, we can obtain an expression for $f(\mathbf{k}', \mathbf{k})$ by making the substitution

$$\psi(\mathbf{r}) \rightarrow \phi(\mathbf{r}) = \frac{\exp(\,i\,\mathbf{k}\cdot\mathbf{r})}{(2\pi)^{3/2}}. \tag{7.35}$$

This is called the *Born approximation*.

$$\begin{aligned} f(\mathbf{k}', \mathbf{k}) &= -\frac{(2\pi)^2\, m}{\hbar^2} \int \frac{\exp(-i\,\mathbf{k}'\cdot\mathbf{r}')}{(2\pi)^{3/2}}\, V(\mathbf{r}')\, \psi(\mathbf{r}')\, d^3\mathbf{r}' \\ &= -\frac{(2\pi)^2\, m}{\hbar^2} \langle \mathbf{k}'|H_1|\psi\rangle. \end{aligned}$$

The Born approximation yields

$$f(\mathbf{k}', \mathbf{k}) \simeq -\frac{m}{2\pi\,\hbar^2} \int \exp\,[\,i\,(\mathbf{k} - \mathbf{k}')\cdot\mathbf{r}']\, V(\mathbf{r}')\, d^3\mathbf{r}'. \tag{7.36}$$

$$E_{sc} \cong C E_0 \frac{\exp[ikR]}{R} \underbrace{\int_{\text{all crystal}} n(\boldsymbol{\rho}) \exp\left[-i\Delta\mathbf{k}.\boldsymbol{\rho}\right] d\boldsymbol{\rho}}_{\text{scattering amplitude}}$$

Thus, $f(\mathbf{k}', \mathbf{k})$ is proportional to the Fourier transform of the scattering potential $V(\mathbf{r})$ with respect to the wave-vector $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$.

For a spherically symmetric potential,

$$f(\mathbf{k}', \mathbf{k}) \simeq -\frac{m}{2\pi\hbar^2}\iiint \exp(i\,q\,r'\cos\theta')\,V(r')\,r'^2 dr'\,\sin\theta'\,d\theta'\,d\phi', \qquad (7.37)$$

giving

$$f(\mathbf{k}', \mathbf{k}) \simeq -\frac{2\,m}{\hbar^2\,q}\int_0^\infty r'\,V(r')\sin(q\,r')\,dr'. \qquad (7.38)$$

Note that $f(\mathbf{k}', \mathbf{k})$ is just a function of q for a spherically symmetric potential. It is easily demonstrated that

$$q \equiv |\mathbf{k} - \mathbf{k}'| = 2\,k\,\sin(\theta/2), \qquad (7.39)$$

where θ is the angle subtended between the vectors $\mathbf{k}$ and $\mathbf{k}'$. In other words, θ is the angle of scattering. Recall that the vectors $\mathbf{k}$ and $\mathbf{k}'$ have the same length by energy conservation.

$$I = \int_0^\pi \sin\theta d\theta \exp\left(-iqr\cos\theta\right)$$

$$I = \left[\frac{1}{qr}2\sin qr\right]$$

$$I = \int_0^{\pi} \sin\theta d\theta \exp(-iqr\cos\theta)$$

$$-iqr\cos\theta = u \Rightarrow iqr\sin\theta d\theta = du$$

$$I = \int \frac{du}{iqr}\exp(u)$$

$$I = \left[\frac{1}{iqr}\exp(-iqr\cos\theta)\right]_0^{\pi}$$

$$I = \frac{1}{iqr}\left[\exp(iqr) - \exp(-iqr)\right]$$

$$I = \left[\frac{1}{iqr}2i\sin qr\right]$$

$$I = \left[\frac{1}{qr}2\sin qr\right]$$

Consider scattering by a Yukawa potential

$$V(r) = \frac{V_0 \exp(-\mu r)}{\mu r}, \tag{7.40}$$

where V_0 is a constant and $1/\mu$ measures the “range” of the potential. It follows from Eq. (7.38) that

$$f(\mathbf{k}', \mathbf{k}) \simeq -\frac{2m}{\hbar^2 q} \int_0^\infty r' V(r') \sin(q r') \, dr'. \tag{7.38}$$

$$f(\theta) = -\frac{2 m V_0}{\hbar^2 \mu} \frac{1}{q^2 + \mu^2}, \tag{7.41}$$

since

$$\int_0^\infty \exp(-\mu r') \sin(q r') \, dr' = \frac{q}{\mu^2 + q^2}. \tag{7.42}$$

Thus, in the Born approximation, the differential cross-section for scattering by a Yukawa potential is

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2 m V_0}{\hbar^2 \mu}\right)^2 \frac{1}{[2 k^2 (1 - \cos\theta) + \mu^2]^2}, \tag{7.43}$$

given that

$$q^2 = 4 k^2 \sin^2(\theta/2) = 2 k^2 (1 - \cos\theta). \tag{7.44}$$

The Yukawa potential reduces to the familiar Coulomb potential as $\mu \to 0$, provided that $V_0/\mu \to Z Z' e^2/4\pi\epsilon_0$. In this limit the Born differential cross-section becomes

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2\,m\,Z\,Z'\,e^2}{4\pi\epsilon_0\,\hbar^2}\right)^2 \frac{1}{16\,k^4\,\sin^4(\theta/2)}. \tag{7.45}$$

Recall that $\hbar\,k$ is equivalent to $|\mathbf{p}|$, so the above equation can be rewritten

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{Z\,Z'\,e^2}{16\pi\epsilon_0\,E}\right)^2 \frac{1}{\sin^4(\theta/2)}, \tag{7.46}$$

where $E = p^2/2\,m$ is the kinetic energy of the incident particles. Equation (7.46) is the classical Rutherford scattering cross-section formula.

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2mV_0}{\hbar^2\mu}\right)^2 \frac{1}{|2k^2(1-\cos\theta)+\mu^2|^2}$$

$$\mu = 0, \frac{V_0}{\mu} = \frac{ZZ'e^2}{4\pi\varepsilon_0\hbar^2}$$

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2m}{\hbar^2}\frac{ZZ'e^2}{4\pi\varepsilon_0\hbar^2}\right)^2 \frac{1}{|4k^2\sin^2\theta|^2}$$

$$E^2 = (\frac{p^2}{2m})^2 = (\frac{\hbar^2k^2}{2m})^2 = \frac{\hbar^4k^4}{(2m)^2}$$

The Born approximation is valid provided that $\psi(\mathbf{r})$ is not too different from $\phi(\mathbf{r})$ in the scattering region. It follows, from Eq. (7.17), that the condition for $\psi(\mathbf{r}) \simeq \phi(\mathbf{r})$ in the vicinity of $\mathbf{r} = 0$ is

$$\left| \frac{m}{2\pi\hbar^2} \int \frac{\exp(i\,k\,r')}{r'}\, V(\mathbf{r}')\, d^3\mathbf{r}' \right| \ll 1. \tag{7.47}$$

Consider the special case of the Yukawa potential. At low energies, (*i.e.*, $k \ll \mu$) we can replace $\exp(i\,k\,r')$ by unity, giving

$$\psi^{\pm}(\mathbf{r}) = \phi(\mathbf{r}) - \frac{2\,m}{\hbar^2} \int \frac{\exp(\pm i\,k\,|\mathbf{r}-\mathbf{r}'|)}{4\pi\,|\mathbf{r}-\mathbf{r}'|}\, V(\mathbf{r}')\, \psi^{\pm}(\mathbf{r}')\, d^3\mathbf{r}'. \tag{7.17}$$

$$\frac{2\,m}{\hbar^2}\frac{|V_0|}{\mu^2} \ll 1 \tag{7.48}$$

as the condition for the validity of the Born approximation. The condition for the Yukawa potential to develop a bound state is

$$\frac{2\,m}{\hbar^2}\frac{|V_0|}{\mu^2} \geq 2.7, \tag{7.49}$$

where V_0 is negative. Thus, if the potential is strong enough to form a bound state then the Born approximation is likely to break down. In the high-k limit, Eq. (7.47) yields

$$\frac{2\,m}{\hbar^2}\frac{|V_0|}{\mu\,k} \ll 1. \tag{7.50}$$

This inequality becomes progressively easier to satisfy as k increases, implying that the Born approximation is more accurate at high incident particle energies.

7.4 Partial waves

$$f(\mathbf{k}', \mathbf{k}) \simeq -\frac{2m}{\hbar^2 q}\int_0^\infty r' V(r') \sin(q r')\, dr'. \tag{7.38}$$

We can assume, without loss of generality, that the incident wave-function is characterized by a wave-vector $\mathbf{k}$ which is aligned parallel to the z-axis. The scattered wave-function is characterized by a wave-vector $\mathbf{k}'$ which has the same magnitude as $\mathbf{k}$, but, in general, points in a different direction. The direction of $\mathbf{k}'$ is specified by the polar angle θ (*i.e.*, the angle subtended between the two wave-vectors), and an azimuthal angle φ about the z-axis. Equation (7.38) strongly suggests that for a spherically symmetric scattering potential [*i.e.*, $V(\mathbf{r}) = V(r)$] the scattering amplitude is a function of θ only:

$$f(\theta, \varphi) = f(\theta). \tag{7.51}$$

It follows that neither the incident wave-function,

$$\phi(\mathbf{r}) = \frac{\exp(\,i\,k\,z\,)}{(2\pi)^{3/2}} = \frac{\exp(\,i\,k\,r\cos\theta\,)}{(2\pi)^{3/2}}, \tag{7.52}$$

nor the total wave-function, $q = |\mathbf{k} - \mathbf{k}'| = (2k^2 - 2\mathbf{k}.\mathbf{k}')^{1/2} = (2k^2 - 2\mathbf{k}.\mathbf{k}')^{1/2} = (2k^2 - 2k^2\cos\theta)^{1/2} = k(2(1-\cos\theta))^{1/2} = 2k\sin\theta/2$

$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\left[\exp(\,i\,k\,r\cos\theta\,) + \frac{\exp(\,i\,k\,r\,)\,f(\theta)}{r}\right], \tag{7.53}$$

depend on the azimuthal angle φ.

Outside the range of the scattering potential, both $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ satisfy the free space Schrödinger equation

$$(\nabla^2 + k^2)\,\psi = 0. \tag{7.54}$$

What is the most general solution to this equation in spherical polar coordinates which does not depend on the azimuthal angle φ? Separation of variables yields

$$\psi(r,\theta) = \sum_l R_l(r)\,P_l(\cos\theta), \tag{7.55}$$

since the Legendre functions $P_l(\cos\theta)$ form a complete set in θ-space. The Legendre functions are related to the spherical harmonics introduced in Sect. 5 via

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2\,l+1}}\,Y_l^0(\theta,\varphi). \tag{7.56}$$

Equations (7.54) and (7.55) can be combined to give

$$r^2\frac{d^2R_l}{dr^2} + 2\,r\frac{dR_l}{dr} + [k^2\,r^2 - l\,(l+1)]R_l = 0. \tag{7.57}$$

$$r^2\frac{d^2R_l}{dr^2}+2\,r\frac{dR_l}{dr}+[k^2\,r^2-l\,(l+1)]R_l=0. \tag{7.57}$$

The two independent solutions to this equation are called a spherical Bessel function, $j_l(k\,r)$, and a Neumann function, $\eta_l(k\,r)$. It is easily demonstrated that

$$j_l(y) \;=\; y^l\left(-\frac{1}{y}\frac{d}{dy}\right)^l\frac{\sin y}{y}, \tag{7.58}$$

$$\eta_l(y) \;=\; -y^l\left(-\frac{1}{y}\frac{d}{dy}\right)^l\frac{\cos y}{y}. \tag{7.59}$$

Note that spherical Bessel functions are well-behaved in the limit $y\to 0$, whereas Neumann functions become singular. The asymptotic behaviour of these functions in the limit $y\to\infty$ is

$$j_l(y) \;\to\; \frac{\sin(y-l\,\pi/2)}{y}, \tag{7.60}$$

$$\eta_l(y) \;\to\; -\frac{\cos(y-l\,\pi/2)}{y}. \tag{7.61}$$

$$j_l(\rho) = (-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \left(\frac{\sin\rho}{\rho}\right)$$

$$n_l(\rho) = -(-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \left(\frac{\cos\rho}{\rho}\right)$$

چند تابع اول آنها در زیر نوشته شده اند:

$$j_0(\rho) = \frac{\sin\rho}{\rho}$$

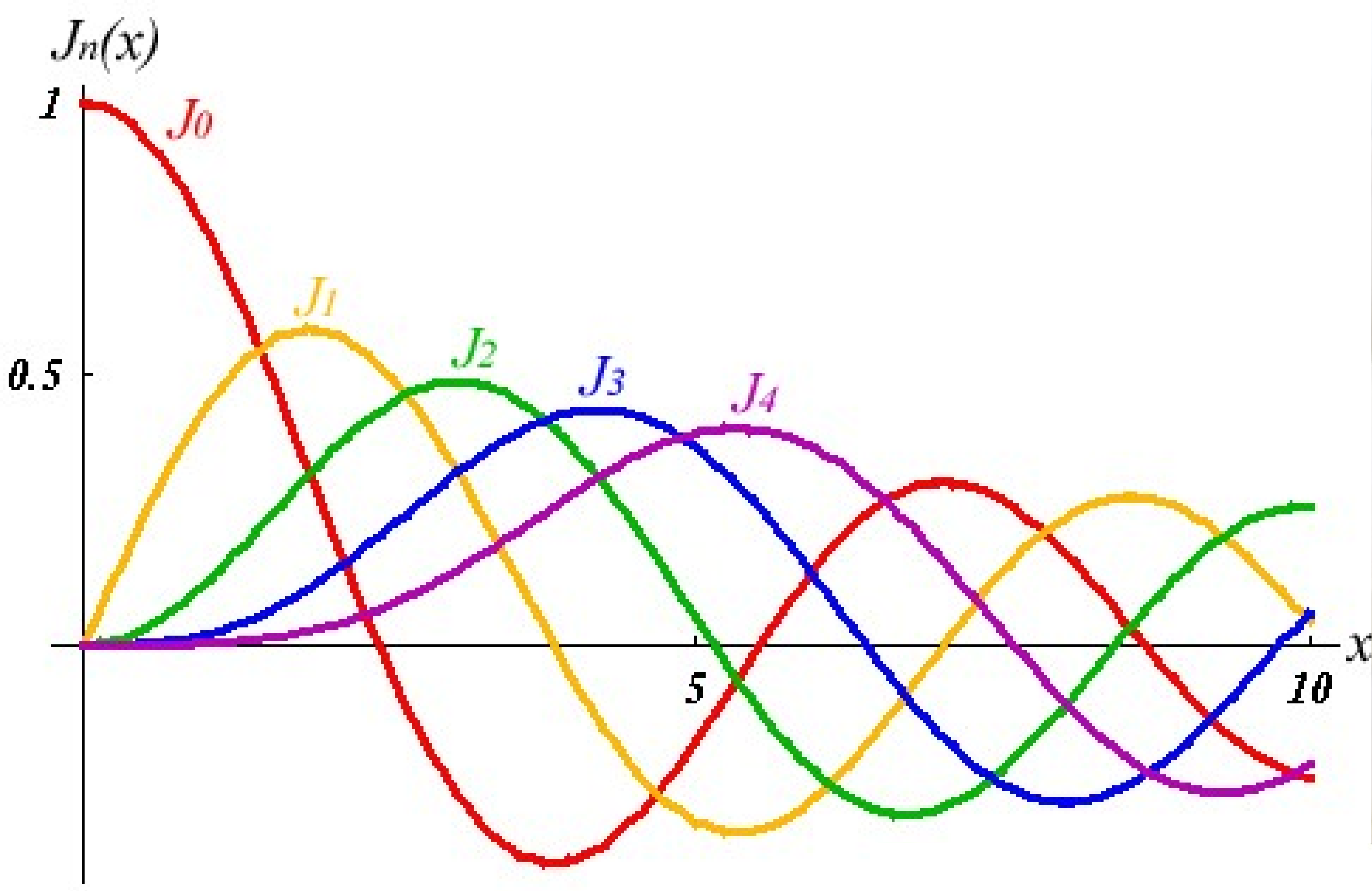
$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}$$

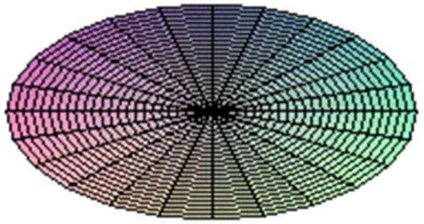
$$j_2(\rho) = \left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\sin\rho - \frac{3}{\rho^2}\cos\rho$$

$$n_0(\rho) = -\frac{\cos\rho}{\rho}$$

$$n_1(\rho) = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho}$$

$$n_2(\rho) = -\left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\cos\rho - \frac{3}{\rho^2}\sin\rho$$


$J_n(x)$
1
0.5
J_0
J_1
J_2
J_3
J_4
5
10
x



We can write

$$\exp(\,i\,k\,r\cos\theta) = \sum_l a_l\, j_l(k\,r)\, P_l(\cos\theta), \tag{7.62}$$

where the a_l are constants. Note there are no Neumann functions in this expansion, because they are not well-behaved as $r \to 0$. The Legendre functions are orthonormal,

$$\int_{-1}^{1} P_n(\mu)\, P_m(\mu)\, d\mu = \frac{\delta_{nm}}{n+1/2}, \tag{7.63}$$

so we can invert the above expansion to give

$$a_l\, j_l(k\,r) = (l+1/2)\int_{-1}^{1} \exp(\,i\,k\,r\,\mu)\, P_l(\mu)\, d\mu. \tag{7.64}$$

It is well-known that

$$j_l(y) = \frac{(-i)^l}{2}\int_{-1}^{1} \exp(\,i\,y\,\mu)\, P_l(\mu)\, d\mu, \tag{7.65}$$

where $l = 0, 1, 2, \cdots$ [see Abramowitz and Stegun (Dover, New York NY, 1965), Eq. 10.1.14]. Thus,

$$a_l = i^l\,(2\,l + 1), \tag{7.66}$$

giving

$$\exp(\,i\,k\,r\cos\theta) = \sum_l i^l\,(2\,l + 1)\,j_l(k\,r)\,P_l(\cos\theta). \tag{7.67}$$

The above expression tells us how to decompose a plane-wave into a series of spherical-waves (or "partial waves").

$$\exp(ikr\cos\theta) = \sum_{l=0}^{\infty}(2l+1)i^{l}\,j_l(kr)P_l(\cos\theta)$$

برای اثبات به طریق زیر عمل می کنیم:

$(J_l(kr)Y_{lm}(\theta, \phi)$ یک پایه کامل است، پس می توان هر تابع موج $\psi(r)$ را بر حسب آنها بسط داد:

$$\Psi(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\,j_l(kr)Y_{lm}(\theta,\varphi)$$

حال فرض کنید که تابع موج $\psi(r)$ پاسخ معادله هلمهولتز $\psi(r)$ (∇^2+k^2)باشد، یعنی $\psi(r)$ $=\exp(ik.r)$ باشد. پس:

$$\exp(i\mathbf{k}.\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\,j_l(kr)Y_{lm}(\theta,\varphi)$$

برای سادگی فرض کنید که پرتوی فرودی در امتداد z باشد:

$$\mathbf{k} = k\hat{\mathbf{z}}$$

$$\exp(i\mathbf{k}.\mathbf{r}) = \exp(ikr\cos\theta)$$

ϕ از مساله حذف می شود

$$m = 0$$

$$\exp(ikr\cos\theta) = \sum_{l=0}^{\infty} A_{l0}\, j_l(kr) Y_{l0}(\theta,\varphi)$$

$$Y_{l,m}(\theta,\varphi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$$

$$Y_{l,0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

$$\exp(ikz) = \sum_{l=0}^{\infty} A_l\, j_l(kr) P_l(\cos\theta).\sqrt{\frac{2l+1}{4\pi}}$$

حال می توان $A_l j_l(kr)$ را به دست آورد:کافی است که طرفین را در $P_{l'}(\cos\theta)$ ضرب و از طرفین انتگرال بگیریم.

$$\int \exp(ikr\cos\theta)P_{l'}(\cos\theta)d(\cos\theta)$$

$$=\sum_{l=0}^{\infty} A_l j_l(kr)\sqrt{\frac{2l+1}{4\pi}}\int P_l(\cos\theta)P_{l'}(\cos\theta)d(\cos\theta)$$

$$\frac{1}{2}\int P_l(\cos\theta)P_{l'}(\cos\theta)d(\cos\theta)=\frac{\delta_{ll'}}{2l+1}$$

$$\int \exp(ikr\cos\theta)P_{l'}(\cos\theta)d(\cos\theta)=A_{l'}j_{l'}(kr)\sqrt{\frac{2l'+1}{4\pi}}\frac{2}{2l'+1}$$

$$A_l j_l(kr)=\sqrt{\pi(2l+1)}\int \exp(ikr\cos\theta)P_l(\cos\theta)d(\cos\theta)$$

$$A_l j_l(kr) = \sqrt{\pi(2l+1)} \int_{-1}^{1} \exp(ikr\cos\theta) P_l(\cos\theta) d(\cos\theta)$$

عبارت فوق را محاسبه نمایید (راهنمایی: از کتاب آرفکن می توانید استفاده کنید)

$$\int_{-1}^{1} \exp(ikr\cos\theta) P_l(\cos\theta) d(\cos\theta) = 2i^l j_l(kr)$$

$$A_l j_l(kr) = \sqrt{\pi(2l+1)}\, 2i^l j_l(kr)$$

$$A_l = \sqrt{\pi(2l+1)}\, 2i^l$$

$$\exp(ikr\cos\theta) = \sum_{l=0}^{\infty} \sqrt{\pi(2l+1)}\, 2i^l j_l(kr) P_l(\cos\theta) . \sqrt{\frac{2l+1}{4\pi}}$$

$$\exp(ikr\cos\theta) = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos\theta)$$

$$\Psi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} j_l(kr) Y_{lm}(\theta,\varphi)$$

برای موج فرودی چون مبدا را شامل می شد
قبلا نوشتیم (نیومن را حذف کرده بودیم)

The most general solution for the total wave-function outside the scattering region is

$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \sum_l [A_l\, j_l(k\,r) + B_l\, \eta_l(k\,r)]\, P_l(\cos\theta), \tag{7.68}$$

$j_l(\rho) \approx \frac{1}{\rho}\sin(\rho - l\frac{\pi}{2})$ $\quad$ $n_l(\rho) \approx -\frac{1}{\rho}\cos(\rho - l\frac{\pi}{2})$ $\quad$ رفتار مجانبی $\rho>>1$

where the A_l and B_l are constants. Note that the Neumann functions are allowed to appear in this expansion, because its region of validity does not include the origin. In the large-r limit, the total wave-function reduces to

$$\psi(\mathbf{r}) \simeq \frac{1}{(2\pi)^{3/2}} \sum_l \left[A_l \frac{\sin(k\,r - l\,\pi/2)}{k\,r} - B_l \frac{\cos(k\,r - l\,\pi/2)}{k\,r}\right] P_l(\cos\theta), \tag{7.69}$$

where use has been made of Eqs. (7.60)–(7.61). The above expression can also be written

$$\psi(\mathbf{r}) \simeq \frac{1}{(2\pi)^{3/2}} \sum_l C_l \frac{\sin(k\,r - l\,\pi/2 + \delta_l)}{k\,r} P_l(\cos\theta), \tag{7.70}$$

where the sine and cosine functions have been combined to give a sine function which is phase-shifted by δ_l.

$$\psi(\mathbf{r}) \simeq \frac{1}{(2\pi)^{3/2}} \sum_l C_l \frac{\sin(k\,r - l\,\pi/2 + \delta_l)}{k\,r} P_l(\cos\theta), \tag{7.70}$$

Equation (7.70) yields

$$\psi(\mathbf{r}) \simeq \frac{1}{(2\pi)^{3/2}} \sum_l C_l \frac{\exp[\,i\,(k\,r - l\,\pi/2 + \delta_l)] - \exp[-i\,(k\,r - l\,\pi/2 + \delta_l)]}{2\,i\,k\,r} \times P_l(\cos\theta), \tag{7.71}$$

which contains both incoming and outgoing spherical-waves. What is the source of the incoming waves? Obviously, they must be part of the large-r asymptotic expansion of the incident wave-function. In fact, it is easily seen that

$$\phi(\mathbf{r}) \simeq \frac{1}{(2\pi)^{3/2}} \sum_l i^l\,(2l+1)\, \frac{\exp[\,i\,(k\,r - l\,\pi/2)] - \exp[-i\,(k\,r - l\,\pi/2)]}{2\,i\,k\,r} \times P_l(\cos\theta) \tag{7.72}$$

in the large-r limit. Now, Eqs. (7.52) and (7.53) give

$$(2\pi)^{3/2}[\psi(\mathbf{r}) - \phi(\mathbf{r})] = \frac{\exp(\,i\,k\,r)}{r} f(\theta). \tag{7.73}$$

$$\phi(\mathbf{r}) = \frac{\exp(\,i\,k\,z)}{(2\pi)^{3/2}} = \frac{\exp(\,i\,k\,r\cos\theta)}{(2\pi)^{3/2}}, \tag{7.52}$$

$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \left[\exp(\,i\,k\,r\cos\theta) + \frac{\exp(\,i\,k\,r)\,f(\theta)}{r}\right], \tag{7.53}$$

$$(2\pi)^{3/2}[\psi(\mathbf{r}) - \phi(\mathbf{r})] = \frac{\exp(\,i\,k\,r)}{r}\,f(\theta). \tag{7.73}$$

Note that the right-hand side consists only of an outgoing spherical wave. This implies that the coefficients of the incoming spherical waves in the large-r expansions of $\psi(\mathbf{r})$ and $\phi(\mathbf{r})$ must be equal. It follows from Eqs. (7.71) and (7.72) that

$$C_l = (2\,l+1)\,\exp[i\,(\delta_l + l\,\pi/2)]. \tag{7.74}$$

Thus, Eqs. (7.71)–(7.73) yield

$$f(\theta) = \sum_{l=0}^{\infty}(2\,l+1)\,\frac{\exp(\,i\,\delta_l)}{k}\,\sin\delta_l\,P_l(\cos\theta). \tag{7.75}$$

Clearly, determining the scattering amplitude $f(\theta)$ via a decomposition into partial waves (*i.e.*, spherical-waves) is equivalent to determining the phase-shifts δ_l.

$$\psi(\mathbf{r}) \;\simeq\; \frac{1}{(2\pi)^{3/2}}\sum_l C_l\,\frac{\exp[i\,(k\,r - l\,\pi/2 + \delta_l)] - \exp[-i\,(k\,r - l\,\pi/2 + \delta_l)]}{2\,i\,k\,r} \times P_l(\cos\theta), \tag{7.71}$$

Optical Theorem

$$\operatorname{Im} f(\theta = 0) = \frac{k\sigma_{tot}}{4\pi} \tag{7.3.1}$$

where

$$f(\theta = 0) \equiv f(\mathbf{k}, \mathbf{k}), \tag{7.3.2}$$

the setting of $\mathbf{k}' \equiv \mathbf{k}$ *imposes scattering in the forward direction, and*

$$\sigma_{tot} \equiv \int \frac{d\sigma}{d\Omega}\, d\Omega. \tag{7.3.3}$$

$$f(\theta) = \sum_{l=0}^{\infty} (2\,l+1)\,\frac{\exp(\,i\,\delta_l)}{k}\,\sin\delta_l\,P_l(\cos\theta). \tag{7.75}$$

The differential scattering cross-section $d\sigma/d\Omega$ is simply the modulus squared of the scattering amplitude $f(\theta)$. The total cross-section is given by

$$\begin{aligned}\sigma_{total} &= \int |f(\theta)|^2\, d\Omega\\ &= \frac{1}{k^2}\oint d\varphi \int_{-1}^{1} d\mu \sum_{l}\sum_{l'} (2\,l+1)\,(2\,l'+1)\exp[\,i\,(\delta_l - \delta_{l'}]\\ &\quad\times \sin\delta_l\,\sin\delta_{l'}\,P_l(\mu)\,P_{l'}(\mu),\end{aligned} \tag{7.76}$$

where $\mu = \cos\theta$. It follows that

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) \frac{\exp(i\,\delta_l)}{k} \sin\delta_l\, P_l(\cos\theta). \quad (7.75)$$

$$\mathrm{Im}\, f(\theta) = \sum_l (2l+1) \frac{\sin\delta_l}{k} \sin\delta_l P_l(\cos\theta)$$

$$\mathrm{Im}\, f(0) = \sum_l (2l+1) \frac{\sin^2\delta_l}{k}$$

$$\sigma_{\mathrm{total}} = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2\delta_l, \tag{7.77}$$

where use has been made of Eq. (7.63). A comparison of this result with Eq. (7.75) yields

$$\int_{-1}^{1} P_n(\mu)\, P_m(\mu)\, d\mu = \frac{\delta_{nm}}{n+1/2}, \quad (7.63)$$

$$\sigma_{\mathrm{total}} = \frac{4\pi}{k} \mathrm{Im}\,[f(0)], \tag{7.78}$$

since $P_l(1) = 1$. This result is known as the *optical theorem*. It is a reflection of the fact that the very existence of scattering requires scattering in the forward ($\theta = 0$) direction in order to interfere with the incident wave, and thereby reduce the probability current in this direction.

It is usual to write

$$\sigma_{\mathrm{total}} = \sum_{l=0}^{\infty} \sigma_l, \tag{7.79}$$

where

$$\sigma_l = \frac{4\pi}{k^2} (2l+1) \sin^2\delta_l \tag{7.80}$$

is the lth partial cross-section: *i.e.*, the contribution to the total cross-section from the lth partial wave. Note that the maximum value for the lth partial cross-section occurs when the phase-shift δ_l takes the value $\pi/2$.