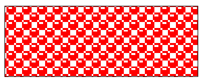
# Examples

We consider, first of all, the problem of a classical ideal gas composed of monatomic particles.



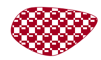
Ideal Gas

In the microcanonical ensemble, the volume ω of the phase space accessible to the representative points of the (member) systems is given by

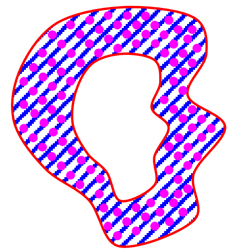
$$\omega = \int' d\omega \equiv \int' (d^{3N}q\, d^{3N}p)$$

where the integrations are restricted by the conditions that (i) the particles of the system are confined in physical space to volume V, and (ii) the total energy of the system lies between the limits (E-1/2Δ) and (E+1/2Δ).

i) V = cte.



ii) $(E - \frac{1}{2}\Delta) \leq H(q,p) \leq (E + \frac{1}{2}\Delta)$



Since the Hamiltonian in this case is a function of the $p_i$ alone, integrations over the $q_i$ can be carried out straightforwardly; these give a factor of $V^N$.

$$\omega = \int\limits_{(E-1/2\Delta)\leq H(q,p)\leq(E+1/2\Delta)} \dots \int d^{3N}q\, d^{3N}p = \int\limits_{(E-1/2\Delta)\leq\sum_{i=1}^{3N}\frac{p_i^2}{2m}\leq(E+1/2\Delta)} \dots \int d^{3N}q\, d^{3N}p = V^N \int\limits_{(E-1/2\Delta)\leq\sum_{i=1}^{3N}\frac{p_i^2}{2m}\leq(E+1/2\Delta)} \dots \int d^{3N}p$$

The remaining integral can be written as follows:

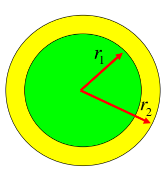
$$\omega = V^N \int\limits_{2m(E-1/2\Delta)\leq\sum_{i=1}^{3N}p_i^2\leq 2m(E+1/2\Delta)} \dots \int d^{3N}y$$

which is equal to the volume of a $3N$-dimensional hypershell, bounded by hyper-spheres of radii

$$\sqrt{[2m(E+\tfrac{1}{2}\Delta)]} \quad \text{and} \quad \sqrt{[2m(E-\tfrac{1}{2}\Delta)]}.$$

For $\Delta \ll E$, this is given by the thickness of the shell, which is almost equal to $\Delta(m/2E)^{1/2}$, multiplied by the surface area of a $3N$-dimensional hypersphere of radius $\sqrt{(2mE)}$. By eqn. (7) of Appendix C, we obtain for this integral

$$\omega = V^N \sqrt{\frac{m}{2E}}\Delta S_{3N}\left(\sqrt{2mE}\right)$$




$$r_2 = \sqrt{2m(E+1/2\Delta)} = \sqrt{2mE\left(1+\frac{1}{2}\frac{\Delta}{E}\right)} \cong \sqrt{2mE}\left(1+\frac{1}{4}\frac{\Delta}{E}\right)$$

$$r_1 = \sqrt{2m(E-1/2\Delta)} = \sqrt{2mE\left(1-\frac{1}{2}\frac{\Delta}{E}\right)} \cong \sqrt{2mE}\left(1-\frac{1}{4}\frac{\Delta}{E}\right)$$

$$r_2 - r_1 = \sqrt{2mE}\left(\frac{\Delta}{2E}\right) = \sqrt{\frac{2mE}{4E^2}}(\Delta) = \sqrt{\frac{m}{2E}}\Delta$$

$$V = 4/3\pi\left\{(2m(E+1/2\Delta))^{3/2} - (2m(E-1/2\Delta))^{3/2}\right\}$$

$$a^3 - b^3 = (a-b)(a^2+b^2+ab)$$

$$V = 4/3\pi\left\{\sqrt{2m(E+1/2\Delta)} - \sqrt{2m(E-1/2\Delta)}\right\}\times$$
$$\left\{(2m(E+1/2\Delta)) + (2m(E-1/2\Delta)) + \sqrt{2m(E+1/2\Delta)}\sqrt{2m(E-1/2\Delta)}\right\}$$

$$V = 4/3\pi\left\{\Delta\sqrt{\frac{m}{2E}}\right\}\times\{4mE+2mE\}$$

$$V = 4/3\pi\left\{\Delta\sqrt{\frac{m}{2E}}\right\}\{6mE\}$$

$$V = \Delta\sqrt{\frac{m}{2E}}4/3\pi\times 6mE = \Delta\sqrt{\frac{m}{2E}}4\pi\times 2mE$$

$$V = \Delta\sqrt{\frac{m}{2E}}4\pi\left(\sqrt{2mE}\right)^2$$

☺

$$V_{3N} = \frac{\pi^{3N/2}}{(3N/2)!}R^{3N} \longrightarrow S_{3N} = \frac{3N\pi^{3N/2}}{(3N/2)!}R^{3N-1}$$

$$\omega = V^N \sqrt{\frac{m}{2E}}\Delta\frac{2\pi^{3N/2}}{(3N/2-1)!}(2mE)^{(3N/2-1)}$$

$$= V^N \sqrt{\frac{m}{2E}}\Delta\frac{2\pi^{3N/2}}{(3N/2-1)!}(2mE)^{(3N/2)}\frac{1}{\sqrt{2mE}}$$

$$= V^N \frac{\Delta}{E}\frac{(2\pi mE)^{3N/2}}{(3N/2-1)!}$$

## Second way for deriving w

$$\omega = V^N \frac{d\Phi}{dE}\delta E$$

$$\Phi(E+\delta E) - \Phi(E) = \frac{d\Phi}{dE}\delta E$$

$$\Phi(E) = V_{3N}(\sqrt{2mE}) = \frac{\pi^{3N/2}}{(3N/2)!}(2mE)^{3N/2}$$

$$\delta E = (E + 1/2\Delta) - (E - 1/2\Delta) = \Delta$$

$$\Phi(E) = \frac{\pi^{3N/2}}{(3N/2)!}(2mE)^{3N/2}$$

$$\frac{d\Phi(E)}{dE} = \frac{\pi^{3N/2}}{(3N/2)!}(2m)^{3N/2}3N/2E^{3N/2}\frac{1}{E}$$

$$= \frac{\pi^{3N/2}}{(3N/2-1)!}(2mE)^{3N/2}\frac{1}{E}$$

$$\omega = V^N \frac{d\Phi}{dE}\delta E$$

$$\omega = V^N \frac{(2\pi mE)^{3N/2}}{(3N/2-1)!}\frac{\Delta}{E}$$ ☺

$$\omega = \frac{(2\pi mE)^{3N/2}}{(3N/2-1)!}\frac{\Delta}{E}$$

From Chapter one may remember that

$$\Gamma\Big)_{asym.} = \frac{\Delta}{E}\frac{V^N}{h^{3N}}\frac{(2\pi mE)^{3N/2}}{(3N/2-1)!}$$

Therefore

$$\omega_0 = \frac{\omega}{\Gamma}\Bigg)_{asymp.} = h^{3N}$$

Homework: Prove that this is also true for the case of a relativistic gas

Quite generally for the case of $\aleph$ degrees of freedom we have

$$\omega_0 = h^{\aleph}$$

If N=1 then $\aleph$=3N=3, therefore N=1, whence

$$\omega_0 = \frac{\omega}{\Gamma}\Bigg)_{asymp.} = h^3$$

$$\Gamma = \frac{\omega}{\omega_0}\Bigg)_{asymp.} = \frac{1}{h^3}\omega$$

$$\Sigma(p) \approx \frac{1}{h^3}\int\ldots\int_{p\le P} d^3q\,d^3p = \frac{V}{h^3}4/3\pi p^3$$

$$g(p)dp = \frac{d\Sigma(p)}{dp}dp \approx \frac{V}{h^3}4\pi p^2 dp$$

$$\Sigma(\varepsilon) \approx \frac{V}{h^3}4/3\pi(2m\varepsilon)^{3/2}$$

$$a(\varepsilon)d\varepsilon = \frac{d\Sigma(\varepsilon)}{d\varepsilon}d\varepsilon = \frac{V}{h^3}2\pi(2m)^{3/2}\varepsilon^{1/2}d\varepsilon$$